N/V-LIMIT FOR STOCHASTIC DYNAMICS IN CONTINUOUS PARTICLE SYSTEMS

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Abstract. We provide an N/V-limit for the infinite particle, infinite volume stochastic dynamics associated with Gibbs states in continuous particle systems on $\mathbb{R}^d, d \geq 1$. Starting point is an N-particle stochastic dynamic with singular interaction and reflecting boundary condition in a subset $\Lambda \subset \mathbb{R}^d$ with finite volume (Lebesgue measure) $V = |\Lambda| < \infty$. The aim is to approximate the infinite particle, infinite volume stochastic dynamic by the above N-particle dynamic in Λ as $N \to \infty$ and $V \to \infty$ such that $N/V \to \rho$, where ρ is the particle density. First we derive an improved Ruelle bound for the canonical correlation functions under an appropriate relation between N and V. Then tightness is shown by using the Lyons–Zheng decomposition. The equilibrium measures of the accumulation points are identified as infinite volume canonical Gibbs measures by an integration by parts formula and the accumulation points themselves are identified as infinite particle, infinite volume stochastic dynamics via the associated martingale problem. Assuming a property closely related to Markov uniqueness and weaker than essential self-adjointness, via Mosco convergence techniques we can identify the accumulation points as Markov processes and show uniqueness. I.e., all accumulation corresponding to one invariant canonical Gibbs measure coincide. The proofs work for general repulsive interaction potentials ϕ of Ruelle type and all temperatures, densities, and dimensions $d \geq 1$, respectively. ϕ may have a nontrivial negative part and infinite range as e.g. the Lennard–Jones potential. Additionally, our result provides as a by-product an approximation of grand canonical Gibbs measures by finite volume canonical Gibbs measures with empty boundary condition.

1. Introduction

The infinite particle, infinite volume stochastic dynamics $(\mathbf{X}(t))_{t \geq 0}$ in continuous particle systems is an infinite dimensional diffusion process having a Gibbs measure μ, e.g. of the type studied by Ruelle in [Rue69], as an invariant measure. Physically, it describes the stochastic dynamics of infinite Brownian particles in $\mathbb{R}^d$, $d \geq 1$, which are interacting via the gradient of a pair-potential ϕ. Since each particle can move through each position in space, the system is called continuous and is used for modelling gas and fluids. For realistic models which can be described by these stochastic dynamics, e.g. suspensions, we refer to [Spo86].

The infinite particle, infinite volume stochastic dynamics takes values in the configuration space

$$\Gamma := \big\{\, \gamma \subset \mathbb{R}^d \mid \#(\gamma \cap \Lambda) < \infty \text{ for each compact } \Lambda \subset \mathbb{R}^d \,\big\},$$

and informally solves the following infinite system of stochastic differential equations:

$$dx(t) = -\beta \sum_{\substack{y(t) \in \mathbf{X}(t) \\ y(t) \neq x(t)}} \nabla\phi(x(t) - y(t))\,dt + \sqrt{2}\,dB^x(t),$$

$$\mathbf{P} \circ \mathbf{X}(0)^{-1} = \mu, \tag{1.1}$$

where $x(t) \in \mathbf{X}(t) \in \Gamma$, $(B^x)_{x \in \gamma}$, $\gamma \in \Gamma$, is a sequence of independent Brownian motions and μ is the invariant measure. The study of such diffusions has been initiated by R. Lang [Lan77] (see also [Shi79]), who considered the case $\phi \in C_0^3(\mathbb{R}^d)$ using finite dimensional approximations and stochastic differential equations. More singular ϕ, which are of particular interest in Physics, as e.g. the Lennard–Jones potential, have been treated by H. Osada, [Osa96], and M. Yoshida, [Yos96] (see also [Tan97], [FRT00] for the hard core case). Osada and Yoshida were the first to use Dirichlet forms for the construction of such processes. However, they could not write down the corresponding generators or martingale problems explicitly, hence could not prove that their processes actually solve (1.1) weakly. This, however, was proved in [AKR98b] by showing an integration by parts formula for the respective Gibbs measures. In [AKR98b], also Dirichlet forms were used and all constructions were designed to work particularly for singular potentials of the above mentioned type. Additionally, an explicit expression for the corresponding generator and martingale problem was provided, which shows that the process in [AKR98b] indeed solves (1.1) in the weak sense.

In this paper, by an approximation through N-particle stochastic dynamics in subsets $\Lambda \subset \mathbb{R}^d$ with finite volume (Lebesgue measure) $V = |\Lambda| < \infty$, we construct weak solutions to (1.1). The approximation is done in terms of the N/V limit, i.e., $N \to \infty$ and $V \to \infty$ such that $N/V \to \rho$, where ρ is the particle density.

The N-particle stochastic dynamics in Λ, $(\mathbf{X}(t))_{t \geq 0}$, takes values in the space of N-point configurations in Λ:

$$\Gamma_\Lambda^{(N)} := \{\gamma \subset \Lambda \mid \#(\gamma) = N\} \subset \Gamma.$$

It solves weakly the following N-system of stochastic differential equations before hitting $\partial(\Gamma_\Lambda^{(N)})$:

$$dx(t) = -\beta \sum_{\substack{y(t) \in \mathbf{X}(t) \\ y(t) \neq x(t)}} \nabla\phi(x(t) - y(t))\,dt + \sqrt{2}\,dB^{x_0}(t),$$

with reflecting boundary condition, $\tag{1.2}$

for sufficiently many initial conditions $\gamma_0 \in \Gamma_\Lambda^{(N)}$. Here $x(t) \in \mathbf{X}(t) \in \Gamma_\Lambda^{(N)}$ and $(B^{x_0})_{x_0 \in \gamma_0}$ are N independent Brownian motions starting in x_0. A weak solution to (1.2) has been constructed in [FG04], see Theorem 4.1. There the authors have used the Dirichlet form approach and their construction works for all dimensions and very general interaction potentials ϕ. Essentially, the interaction potential ϕ only has to have a singularity at the

origin (repulsion) (RP), to be bounded from below (BB) and weakly differentiable (D), see below for a precise definitions.

Note that we are only considering configurations with at most one particle in one position, which is a reasonable assumption for modelling gas and fluids. In such a setting for dimension $d = 1$ this is the first existence result for a solution to (1.2). The essential assumption for this result is the condition (RP) (repulsion of close particles), which is natural from the physical point of view.

Our approach is different from the finite dimensional approximation provided by Lang [Lan77]. There for a fixed subset $\Lambda \subset \mathbb{R}^d$ with finite volume, the finite particle, finite volume dynamics consists of finitely but arbitrarily (for different initial conditions) many interacting particles inside the volume and additionally they are interacting with particles from the complement of Λ. That construction is rather in a grand canonical setting whereas ours is in a canonical one. Thus, we expect the finite particle, finite volume dynamics used in [Lan77] for singular interaction potentials with non-trivial negative part not to have such nice properties as our N-particle stochastic dynamics in Λ. E.g., for determining a spectral gap of their generators, it is much nicer to have a fixed number N of particles in a given volume Λ not interacting with particles in the complement of Λ, than finite but arbitrarily many particles inside Λ interacting with in general infinitely many particles in the complement of Λ.

Our plan for future work is to use our approximation by nice processes to get better knowledge about the infinite volume, infinite particle dynamics. For example we would like to: explore in more detail the structure of the spectrum of its generator and study the problem of essential self-adjointness; construct non-equilibrium infinite particle, infinite volume stochastic dynamics; tackle the Boltzmann–Gibbs principle, see e.g. [Spo86] and [GKLR03]; use our approximation technique to construct solutions to other equations as e.g. the Langevin equations.

The present paper is organized in the following way: In Section 2 we define a metric on the configuration space Γ which is appropriate for our problem. This metric is from the class of metrics on Γ developed in [KK04] and induces the vague topology. Essential for our considerations is that these metrics d make (Γ, d) a Polish space and that relative compact sets w.r.t. the vague topology can be described explicitly (cf. [KK04]).

The concept of canonical Gibbs measures, our assumptions on the interaction potential and a precise definition of the N/V-limit are presented in Section 3. Furthermore, in Theorem 3.2 we prove the first major result of this paper. There we establish a bound for canonical correlation functions analogous to the Ruelle bound for grand canonical correlation functions, see (3.3). In the proof we combine ideas of Ruelle's proof [Rue70] for deriving the Ruelle bound in the grand canonical case with estimates obtained in [DM67]. A major difference in comparison with the grand canonical case is that in the canonical case a right balance between the particle number N and the volume V is necessary. Furthermore, we derive an improved Ruelle bound for canonical correlation functions, see (3.4). This bound enables us to take into account potentials with singularities at the origin, see condition (D) below.

In Section 4 we briefly summarize the construction of the N-particle stochastic dynamics in Λ weakly solving (1.2) provided in [FG04].

The N/V-limit of N-particle, finite volume stochastic dynamics is then derived in Section 5. First, in Theorem 5.1 we prove tightness of the sequence of laws $(\mathbf{P}^{(N)})_{N\in\mathbb{N}}$ of the equilibrium N-particle, finite volume stochastic dynamics in the N/V-limit. Equilibrium stochastic dynamics means that the stochastic dynamics starts with an initial distribution given by the corresponding invariant, finite volume canonical Gibbs measure $\mu^{(N)}$. The proof is split into two lemmas. Lemma 5.2 gives tightness of the corresponding one-dimensional distributions (invariant, finite volume canonical Gibbs measures) $(\mu^{(N)})_{N\in\mathbb{N}}$ and essentially depends on the improved Ruelle bound (3.4) and the description of compact sets provided in [KK04]. In Lemma 5.3 we prove Kolmogorov–Chentsov type estimates for the increments. In the proof we use the well-known Lyons–Zheng decomposition, [LZ88], [LZ94], of the N-particle, finite volume stochastic dynamics and the Burkholder–Davies–Gundy inequalities in order to establish the required estimate of the increments. For this it is important to have sufficiently many functions in the domain of the corresponding Dirichlet form, which is in fact implied by the reflecting boundary condition we impose on the N-particle, finite volume stochastic dynamics. Again, also the improved Ruelle bound is of essential importance.

Then in Theorem 5.9 we prove an integration by parts formula for the accumulation points μ of $(\mu^{(N)})_{N\in\mathbb{N}}$. Together with a characterization theorem provided in [AKR98b] this implies that these μ are infinite volume canonical Gibbs measures.

After that, in Theorem 5.10 we identify the accumulation points $\mathbf{P}$ of $(\mathbf{P}^{(N)})_{N\in\mathbb{N}}$ as solutions of (1.1) in the sense of the associated martingale problem. See also Remark 5.11. In the proof we are using that the N-particle, finite volume stochastic dynamics solves the martingale problem corresponding to (1.2).

From Theorem 5.10 we can not conclude that the accumulation points $\mathbf{P}$ of $(\mathbf{P}^{(N)})_{N\in\mathbb{N}}$ are laws of Markov processes. However, assuming a property closely related to Markov uniqueness and weaker than essential self-adjointness, in Theorem 5.20 we can show Mosco convergence, [Mos94], [KS03], of the quadratic forms (Dirichlet forms) corresponding to convergent subsequences. This implies strong convergence of the associated semi-groups. This convergence, in turn, enables us to identify the accumulation points $\mathbf{P}$ as laws of Markov processes and show uniqueness. I.e., all accumulation points $\mathbf{P}$ corresponding to one invariant canonical Gibbs measure coincide, see Theorem 5.23.

Finally, in Section 6 we apply our results to the problem of equivalence of ensembles. More precisely, as a by-product of the results described above we obtain an approximation of grand canonical Gibbs measures by finite volume canonical Gibbs measures with empty boundary condition, see Theorem 6.1.

The progress achieved in this paper may be summarized by the following list of main results:

- Derivation of an improved Ruelle bound for canonical correlation functions, see Theorem 3.2.
- Tightness of the sequence of laws $(\mathbf{P}^{(N)})_{N\in\mathbb{N}}$ of equilibrium N-particle, finite volume stochastic dynamics in the N/V-limit, see Theorem 5.1.
- Identification of the accumulation points μ of the sequence of finite volume canonical Gibbs measures $(\mu^{(N)})_{N\in\mathbb{N}}$ as infinite volume canonical Gibbs measures via an integration by parts formula, see Theorem 5.9.

- Identification of the accumulation points $\mathbf{P}$ of the sequence of laws $(\mathbf{P}^{(N)})_{N \in \mathbb{N}}$ of equilibrium N-particle, finite volume stochastic dynamics in the N/V-limit as solutions of (1.1) in the sense of the associated martingale problem, see Theorem 5.10. This is the first construction of a solution to (1.1) for $d = 1$ with state space Γ (at most one particle in one position).

Furthermore, when assuming a property closely related to Markov uniqueness and weaker than essential self-adjointness:

- Identification of the accumulation points $\mathbf{P}$ of the sequence of laws $(\mathbf{P}^{(N)})_{N \in \mathbb{N}}$ of equilibrium N-particle, finite volume stochastic dynamics in the N/V-limit as Markov processes and showing uniqueness, see Theorem 5.23.

At the moment we are working on the assumed property and expect to show it soon.

All above results apply to all dimensions $d \geq 1$, temperatures and densities and to physically relevant repulsive (RP) interaction potentials ϕ. Additional assumptions are only a mild temperedness (T) condition (fast enough decay in the long range), that the potential is bounded from below (BB) and a mild differentiability (D) condition. Hence, singularities at the origin, non-trivial negative part, and infinite range are allowed.

Hypotheses on the potential are weakened not for the sake of generality, but in order to cover the physically relevant potentials (as e.g. Lennard–Jones potential).

2. A Polish metric for the configuration space

The configuration space Γ over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all subsets of $\mathbb{R}^d$ which are locally finite:

$$\Gamma := \big\{ \gamma \subset \mathbb{R}^d \mid \#(\gamma_\Lambda) < \infty \text{ for each compact } \Lambda \subset \mathbb{R}^d \big\},$$

where $\#$ denotes the number of elements of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}(\mathbb{R}^d)$, where ε_x is the Dirac measure at x, $\sum_{x \in \varnothing} \varepsilon_x :=$ zero measure, and $\mathcal{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. A metric on $\mathcal{M}(\mathbb{R}^d)$ is given by

$$d_\mathcal{M}(\nu, \mu) := \sum_{k=1}^{\infty} 2^{-k} p_k \left(1 - \exp\left(-|\langle f_k, \nu - \mu \rangle| \right) \right), \qquad \nu, \mu \in \mathcal{M}(\mathbb{R}^d),$$

where $\{f_k \mid k \in \mathbb{N}\} \subset C_c^1(\mathbb{R}^d)$ (space of continuously differentiable functions on $\mathbb{R}^d$ with compact support) is a measure determining class, $(p_k)_{k \in \mathbb{N}}$ a sequence of strictly positive weights bounded by 1, and

$$\langle f, \nu \rangle := \int_{\mathbb{R}^d} f \, d\nu, \quad f \in C_c(\mathbb{R}^d), \ \nu \in \mathcal{M}(\mathbb{R}^d).$$

$\{f_k \mid k \in \mathbb{N}\}$ can be chosen so that $d_\mathcal{M}$ induces the vague topology on $\mathcal{M}(\mathbb{R}^d)$. This metrization is separable and complete, see [Kal75, A 7.7] for the case $\{f_k \mid k \in \mathbb{N}\} \subset C_c(\mathbb{R}^d)$ and $p_k = 1$ for all $k \in \mathbb{N}$.

In $\mathcal{M}(\mathbb{R}^d)$ we consider the subset $\mathcal{R}(\mathbb{R}^d)$ consisting of all $\mathbb{Z}_+ \cup \{\infty\}$-valued Radon measures. Since $\mathcal{R}(\mathbb{R}^d)$ is a closed subset of $\mathcal{M}(\mathbb{R}^d)$ w.r.t. the vague convergence, see [Kal75, A 7.4], also $(\mathcal{R}(\mathbb{R}^d), d_\mathcal{M})$ is a Polish space.

Now our aim is to find a metric on Γ which is Polish. Let $\Phi : (0, \infty) \to [0, \infty)$ be a continuous decreasing function such that $\lim_{t \to 0} \Phi(t) = \infty$; and let $h : \mathbb{R}^d \to (0, 1]$ be a function in $L^1(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$. Furthermore, let $\mathbf{I} = \{I_k \,|\, k \in \mathbb{N}\}$ be a collection of functions from $C_c^1(\mathbb{R}^d)$ such that $I_k : \mathbb{R}^d \to [0, 1]$, $\mathrm{supp} I_k \subset B_k(0)$, and $I_{k+1}(x) = 1$ for all $x \in B_k(0)$, here $B_k(0)$ denotes the closed ball with radius k centered at the origin. Define

$$S^{\Phi, f}(\gamma) := \sum_{\{x, y\} \subset \gamma} \exp(\Phi(|x - y|)) f(x) f(y),$$

where $f : \mathbb{R}^d \to [0, \infty)$ is a continuously differentiable function. For any $k \in \mathbb{N}$ set $h_k := h I_k$. Then for $\gamma, \eta \in \Gamma$ we define the metric

$$d_{\Phi, h}(\gamma, \eta) := d_{\mathcal{M}}(\gamma, \eta) + \sum_{k=1}^{\infty} 2^{-k} q_k \frac{\left| S^{\Phi, h_k}(\gamma) - S^{\Phi, h_k}(\eta) \right|}{1 + |S^{\Phi, h_k}(\gamma) - S^{\Phi, h_k}(\eta)|}, \qquad (2.1)$$

where $(q_k)_{k \in \mathbb{N}}$ is a sequence of strictly positive weights bounded by 1. The following has been proved in [KK04, Theo. 3.5, Prop. 3.1] for $q_k = 1$, $k \in \mathbb{N}$. Easily, the statement generalizes to the present situation.

Proposition 2.1. $(\Gamma, d_{\Phi, h})$ *is a complete and separable metric space. Moreover, the topology on Γ generated by the metric $d_{\Phi, h}$ is equivalent to the vague topology on Γ and the sets*

$$\{\gamma \in \Gamma \,|\, S^{\Phi, h}(\gamma) \leq R\}, \qquad R < \infty,$$

are relative compact subsets w.r.t. the vague topology.

3. Canonical Gibbs measures and an improved Ruelle bound

Let $\Lambda \subset \mathbb{R}^d$. We denote $\Gamma_\Lambda := \{\gamma \in \Gamma \,|\, \gamma \subset \Lambda\}$. For any $N \in \mathbb{N}$ and bounded Borel measurable $\Lambda \subset \mathbb{R}^d$ we define the space of N-point configurations in Λ by

$$\Gamma_\Lambda^{(N)} := \{\gamma \subset \Lambda \,|\, \#(\gamma) = N\} \subset \Gamma_\Lambda.$$

To define more structure on $\Gamma_\Lambda^{(N)}$ we use the following natural mapping

$$\mathrm{sym}^{(N)} : \widetilde{\Lambda^N} \to \Gamma_\Lambda^{(N)}$$

$$\mathrm{sym}^{(N)}((x_1, \ldots, x_N)) := \{x_1, \ldots, x_N\},$$

where

$$\widetilde{\Lambda^N} := \{(x_1, \ldots, x_N) \in \Lambda^N \,|\, x_k \neq x_j \quad \text{if} \quad k \neq j\}.$$

These mappings generate a topology and corresponding Borel σ-algebra on $\Gamma_\Lambda^{(N)}$. Obviously, this σ-algebra coincides with the Borel σ-algebra inherited from Γ equipped with its vague topology. We denote by dx_Λ the Lebesgue measure on Λ. Then the product measure $dx_\Lambda^{\otimes N}$ can be considered on $\widetilde{\Lambda^N}$. Let $dx_\Lambda^{(N)} := dx_\Lambda^{\otimes N} \circ (\mathrm{sym}^{(N)})^{-1}$ be the corresponding measure on $\Gamma_\Lambda^{(N)}$.

A pair potential (without hard core) is a Borel measurable function $\phi \colon \mathbb{R}^d \to \mathbb{R} \cup \infty$ such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. For bounded Borel measurable $\Lambda \subset \mathbb{R}^d$ the potential energy $E_\phi \colon \Gamma_\Lambda \to \mathbb{R}$ in Λ with empty boundary condition is defined by

$$E_\phi(\gamma) := \sum_{\{x,y\} \subset \gamma} \phi(x-y), \qquad \gamma \in \Gamma_\Lambda,$$

where the sum over the empty set is defined to be zero. The interaction energy between two configurations γ and η from Γ_Λ is defined by

$$W_\phi(\gamma, \eta) := \sum_{x \in \gamma,\, y \in \eta} \phi(x-y).$$

Note that

$$E_\phi(\gamma \cup \eta) = E_\phi(\gamma) + W_\phi(\gamma, \eta) + E_\phi(\eta), \qquad \gamma, \eta \in \Gamma_\Lambda.$$

Now we fix our assumptions on ϕ:

(RP): (*Repulsion*) There exists a decreasing continuous function $\Phi : (0, \infty) \to [0, \infty)$ with $\lim_{t \to 0} \Phi(t) t^d = \infty$ and $R_1 > 0$ such that

$$\phi(x) \geq \Phi(|x|) \quad \text{for} \quad |x| \leq R_1.$$

Furthermore, the potential ϕ is bounded from above on $\{x \in \mathbb{R}^d \,|\, r \leq |x| \leq R_1\}$ for all $r > 0$.

(T): (*Temperedness*) The exists $A, R_2 < \infty$ and $\lambda > d$ such that

$$|\phi(x)| \leq A|x|^{-\lambda} \quad \text{for} \quad |x| \geq R_2.$$

(BB): (*Bounded below*) There exist $B \geq 0$ such that

$$\phi(x) \geq -B, \quad \text{for all} \quad x \in \mathbb{R}^d.$$

For every $r = (r^1, \dots, r^d) \in \mathbb{Z}^d$, we define a cube

$$Q(r) := \left\{ x \in \mathbb{R}^d \,\middle|\, r^i - \frac{1}{2} \leq x^i < r^i + \frac{1}{2} \right\}.$$

These cubes form a partition of $\mathbb{R}^d$. For any $\gamma \in \Gamma$, we set $\gamma_r := \gamma_{Q(r)}$, $r \in \mathbb{Z}^d$.

(RP), (T) and (BB) imply that ϕ is superstabile (SS) and lower regular (LR), see [Rue70, Prop. 1.4]. That is:

(SS): (*Superstability*) There exist $D > 0$, $K \geq 0$ such that, if $\gamma \in \Gamma_\Lambda$, where Λ is a finite union of the cubes $Q(r)$, then

$$\sum_{\{x,y\} \subset \gamma} \phi(x-y) \geq \sum_{r \in \mathbb{Z}^d} \left(D \#(\gamma_r)^2 - K \#(\gamma_r) \right).$$

(LR): (*Lower regularity*) There exists a decreasing positive function $\Psi : \mathbb{N} \to [0, \infty)$ such that

$$\sum_{r \in \mathbb{Z}^d} \Psi(|r|_{\max}) < \infty$$

and for any disjoint Λ', Λ'' which are finite unions of the cubes $Q(r)$, we have for $\gamma' \in \Gamma_{\Lambda'}$, $\gamma'' \in \Gamma_{\Lambda''}$:

$$W_\phi(\gamma', \gamma'') \geq - \sum_{r', r'' \in \mathbb{Z}^d} \Psi(|r' - r''|_{\max}) \#(\gamma'_{r'}) \#(\gamma''_{r''}).$$

Here $|\cdot|_{\max}$ denotes the maximum norm on $\mathbb{R}^d$.

Moreover, (T) and (BB) imply

$$J(\beta) := \int_{\mathbb{R}^d} |\exp(-\beta\phi(x)) - 1| \, dx < \infty \tag{3.1}$$

for all $\beta \geq 0$ (dx denotes the Lebesgue measure on $\mathbb{R}^d$). The property (3.1) is also called *integrability* (I) or *regularity*.

On $(\Gamma_\Lambda^{(N)}, \mathcal{B}(\Gamma_\Lambda^{(N)}))$ we consider the canonical N-particle Gibbs measures $\mu_\Lambda^{(N)}$ in Λ with empty boundary condition:

$$\mu_\Lambda^{(N)} := \frac{1}{Z_\Lambda^{(N)}} \exp\left(-\beta E_\phi\right) dx_\Lambda^{(N)},$$

where

$$Z_\Lambda^{(N)} := \int_{\Gamma_\Lambda^{(N)}} \exp\left(-\beta E_\phi\right) dx_\Lambda^{(N)}$$

is the canonical partition function of N particles in Λ. The constant $\beta \geq 0$ is the inverse temperature.

For $1 \leq n \leq N$ the n-order correlation function corresponding to $\mu_\Lambda^{(N)}$ is defined by

$$k_\Lambda^{(n,N)}(x_1, \ldots, x_n) := \frac{N \cdot \ldots \cdot (N - n + 1)}{Z_\Lambda^{(N)}} \int_{\Lambda^{N-n}} \exp\left(-\beta(E_\phi(X \cup Y))\right) dy_\Lambda^{\otimes(N-n)},$$

where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_{N-n}\}$. Furthermore, we define

$$k_\Lambda^{(0,N)} := 1 \quad \text{and} \quad k_\Lambda^{(n,N)} := 0 \quad \text{for} \quad n > N.$$

Let $f^{(n)} : \Lambda^n \to [0, \infty]$ be a symmetric measurable function, $1 \leq n \leq N$, then

$$\int_{\Gamma_\Lambda^{(N)}} \sum_{\{x_1, \ldots, x_n\} \subset \gamma} f^{(n)}(x_1, \ldots, x_n) \, d\mu_\Lambda^{(N)}(\gamma)$$

$$= \frac{1}{n!} \int_{\Lambda^n} f^{(n)}(x_1, \ldots, x_n) k_\Lambda^{(n,N)}(x_1, \ldots, x_n) \, dx_\Lambda^{\otimes n}. \tag{3.2}$$

Definition 3.1. Let $(\Lambda_N)_{N \in \mathbb{N}}$ be a sequence of bounded Borel measurable subsets of $\mathbb{R}^d$ with $|\Lambda_N| > 0$ which exhausts $\mathbb{R}^d$, i.e., for each bounded $\Lambda \subset \mathbb{R}^d$ there exists $N_\Lambda \in \mathbb{N}$ such that $\Lambda \subset \Lambda_N$ for all $N \geq N_\Lambda$. We denote by $|A|$ the Lebesgue measure of a Borel measurable set $A \subset \mathbb{R}^d$. We say that $(\Lambda_N)_{N \in \mathbb{N}}$ has an N/V-limit, if

$$\rho := \lim_{N \to \infty} \frac{N}{|\Lambda_N|}$$

exists in $(0, \infty)$. In this case we call $(\Lambda_N)_{N \in \mathbb{N}}$ a sequence of volumes corresponding to the density $\rho > 0$. Sometimes we use the notation $v_N := N/|\Lambda_N|$.

Theorem 3.2. *Suppose that the conditions* (RP), (T), *and* (BB) *are satisfied and let* $(\Lambda_N)_{N \in \mathbb{N}}$ *be a sequence of volumes corresponding to the density* $\rho > 0$. *Then for large enough* $N_0 \in \mathbb{N}$ *(such that* $|\Lambda_N|$ *is larger than a critical volume for all* $N \geq N_0$*) there exists* $\xi < \infty$ *such that*

$$k_{\Lambda_N}^{(n,N)}(x_1, \ldots, x_n) \leq \xi^n \quad \text{for all} \quad N \geq N_0, \, n \in \mathbb{N}, \, (x_1, \ldots, x_n) \in (\Lambda_N)^n \qquad (3.3)$$

(Ruelle bound). Moreover, for $n \geq 2$ *there exists* $\zeta < \infty$ *such that*

$$k_{\Lambda_N}^{(n,N)}(x_1, \ldots, x_n) \leq \exp\Big(-\frac{2}{n} \sum_{1 \leq i < j \leq n} \beta \phi(x_i - x_j) \Big) \zeta^n$$
$$\text{for all} \quad N \geq N_0, \, n \geq 2, \, (x_1, \ldots, x_n) \in (\Lambda_N)^n \quad (3.4)$$

(improved Ruelle bound).

Proof: The Ruelle bound for grand canonical correlation functions is derived in [Rue70, Prop. 2.6]. Here we adapt that proof to canonical correlations functions. For this, additionally, we need the following estimates for canonical partition functions provided in [DM67, Lem. 3']: For $|\Lambda_N|$ large enough there exists a constant $C_1 < \infty$ such that

$$\frac{Z_{\Lambda_N}^{(n-1)}}{Z_{\Lambda_N}^{(n)}} \leq C_1 \frac{1}{|\Lambda_N|} \quad \text{for all} \quad 1 \leq n \leq N. \qquad (3.5)$$

Note that because $(\Lambda_N)_{N \in \mathbb{N}}$ has an N/V-limit, there exists $C_2 < \infty$ such that

$$v_N \leq C_2 \quad \text{for all} \quad N \in \mathbb{N}. \qquad (3.6)$$

Now we need to introduce some notation from [Rue70]. Let $(l_j)_{j \in \mathbb{N}}$ be an increasing sequence in $\mathbb{N}$. We define

$$[j] := \{r \in \mathbb{Z}^d \,|\, |r|_{\max} \leq l_j\}, \quad V_j := \sum_{r \in [j]} Q(r).$$

Furthermore, let ψ be an increasing function on $\mathbb{N}$ such that

$$\psi \geq 1, \quad \lim_{j \to \infty} \psi(j) = \infty, \quad \text{and} \quad \sum_{r \in \mathbb{Z}^d} \psi(|r|_{\max}) \Psi(|r|_{\max}) < \infty.$$

Define $\psi_j := \psi(l_j)$ and let $P > 0$. Then for each $X \cup Y$, $X = \{x_1, \ldots, x_n\}, Y = \{y_1, \ldots, y_{N-n}\}$ either

$$\sum_{r \in [j]} \#(\{X \cup Y\}_r)^2 \leq \psi_j |V_j| \qquad (3.7)$$

for all $j \geq P$ or there exists a largest $q \geq P$ such that

$$\sum_{r \in [q]} \#(\{X \cup Y\}_r)^2 \geq \psi_q |V_q|. \qquad (3.8)$$

Now let P, the sequence $(l_j)_{j \in \mathbb{N}}$ and function ψ be chosen as in [Rue70, Sect. 2]. Then there exists $C_3 < \infty$ such

$$-W_\phi(\{x_1\}, X \setminus \{x_1\} \cup Y) \leq C_3 \qquad (3.9)$$

for all $X \cup Y$ fulfilling (3.7), see [Rue70, Eq. (2.29)]. On the other hand, for all $X \cup Y$ fulfilling (3.8) there exists $C_4 > 0$ such that

$$
- E_\phi(\{X \cup Y\}_{V_{q+1}}) - W_\phi(\{X \cup Y\}_{V_{q+1}}, \{X \cup Y\}_{V_{q+1}^c})
$$
$$
\leq -\frac{D}{4} \sum_{r \in [q+1]} \#(\{X \cup Y\}_r)^2 - C_4 \psi_{q+1} |V_{q+1}|, \tag{3.10}
$$

where the constant D is as in (SS), see [Rue70, Prop. 2.5].

We prove the assertion by induction. Let us fix $X = \{x_1, \ldots, x_n\}$, $n \geq 1$, and choose the coordinates of $\mathbb{Z}^d$ such that $x_1 \in Q(0)$. Let $S_0 \subset \Lambda_N^{N-n}$ such that $X \cup Y$ fulfills (3.7) for all $(y_1, \ldots, y_{N-n}) \in S_0$ and $S_q \subset \Lambda_N^{N-n}$ such that $X \cup Y$ fulfills (3.8) for all $(y_1, \ldots, y_{N-n}) \in S_p$. Furthermore, we define

$$
C_5 := \max \left\{ \left(\exp(\beta C_3) C_1 + \sum_{q \geq P} \exp\left(-(\beta C_4 \psi_{q+1} - 1)|V_{q+1}| \right) \right) C_2, C_2, 1 \right\}
$$

and assume $\xi \geq C_5$.

Now (3.9) together with (3.5) and (3.6) implies:

$$
\frac{N \cdot \ldots \cdot (N-n+1)}{Z_{\Lambda_N}^{(N)}} \int_{S_0} \exp\left(-\beta E_\phi(X \cup Y) \right) dy_{\Lambda_N}^{\otimes(N-n)}
$$
$$
\leq \frac{N \cdot \ldots \cdot (N-n+1)}{Z_{\Lambda_N}^{(N)}} \exp(\beta C_3) \int_{S_0} \exp\left(-\beta E_\phi(X \setminus \{x_1\} \cup Y) \right) dy_{\Lambda_N}^{\otimes(N-n)}
$$
$$
\leq \exp(\beta C_3) C_1 C_2 k_{\Lambda_N}^{(n-1,N-1)}(x_2, \ldots, x_n) \leq \exp(\beta C_3) C_1 C_2 \xi^{n-1}. \tag{3.11}
$$

In turn, (3.10) together with (3.5) and (3.6) yields:

$$
\frac{N \cdot \ldots \cdot (N-n+1)}{Z_{\Lambda_N}^{(N)}} \int_{S_q} \exp\left(-\beta E_\phi(X \cup Y) \right) dy_{\Lambda_N}^{\otimes(N-n)}
$$
$$
\leq \frac{N \cdot \ldots \cdot (N-n+1)}{Z_{\Lambda_N}^{(N)}} \exp\left(-\beta \frac{D}{4} \sum_{r \in V_{q+1}} \#(\{X \cup Y\}_r)^2 - \beta C_4 \psi_{q+1} |V_{q+1}| \right)
$$
$$
\times \int_{\Lambda_N^{N-n}} \exp\left(-\beta E_\phi(\{X \cup Y\}_{V_{q+1}^c}) \right) dy_{\Lambda_N}^{\otimes(N-n)}
$$
$$
\leq \frac{N \cdot \ldots \cdot (N-n+1)}{Z_{\Lambda_N}^{(N)}} \exp\left(-\beta \frac{D}{4} \sum_{r \in V_{q+1}} \#(\{X \cup Y\}_r)^2 - \beta C_4 \psi_{q+1} |V_{q+1}| \right)
$$
$$
|\Lambda_N|^{\#(Y_{V_{q+1}})} \int_{\Lambda_N^{N-n-\#(Y_{V_{q+1}})}} \exp\left(-\beta E_\phi(\{X \cup Y\}_{V_{q+1}^c}) \right) dy_{\Lambda_N}^{\otimes(N-n-\#(Y_{V_{q+1}}))}
$$
$$
\leq \exp\left(-\beta \frac{D}{4} \sum_{r \in V_{q+1}} \#(\{X \cup Y\}_r)^2 - \beta C_4 \psi_{q+1} |V_{q+1}| \right)
$$
$$
\times (C_1)^{\#(Y_{V_{q+1}})} (C_1|v_N|)^{\#(X_{V_{q+1}})} k_{\Lambda_N}^{(n-\#(X_{V_{q+1}}), \#(Y_{V_{q+1}^c}))}(x_{i_1}, \ldots, x_{i_{n-\#(X_{V_{q+1}})}})
$$

$$\leq \exp\left(-\beta\frac{D}{4}\sum_{r\in V_{q+1}}\#(\{X\cup Y\}_r)^2 + \ln(C_1)\sum_{r\in V_{q+1}}\#(\{X\cup Y\}_r)\right.$$

$$\left. - \beta C_4\psi_{q+1}|V_{q+1}|\right)C_2\xi^{n-1} \leq \exp\left(-(\beta C_4\psi_{q+1}-1)|V_{q+1}|\right)C_2\xi^{n-1}, \quad (3.12)$$

where we used that

$$-\beta\frac{D}{4}\#(\{X\cup Y\}_r)^2 + \ln(C_1)\#(\{X\cup Y\}_r) \leq 1.$$

Finally, summing up (3.11) and (3.12) we get

$$k_{\Lambda_N}^{(n,N)}(x_1,\ldots,x_n) \leq \left(\exp(\beta C_3)C_1 + \sum_{q\geq P}\exp\left(-(\beta C_4\psi_{q+1}-1)|V_{q+1}|\right)\right)C_2\xi^{n-1} \leq \xi^n.$$

The canonical correlation functions fulfill the following Kirkwood–Salsburg type equations:

$$k_{\Lambda_N}^{(n,N)}(x_1,\ldots,x_n) = N\frac{Z_{\Lambda_N}^{(N-1)}}{Z_{\Lambda_N}^{(N)}}\exp\left(-\sum_{2\leq i\leq n}\beta\phi(x_1-x_j)\right)\left(k_{\Lambda_N}^{(n-1,N-1)}(x_2,\ldots,x_n)\right.$$

$$\left. + \sum_{k=1}^{N-n}\frac{1}{k!}\int_{\Lambda_N^k}k_{\Lambda_N}^{(n+k-1,N-1)}(x_2,\ldots,x_n,y_1,\ldots,y_k)\prod_{i=1}^{k}\left(\exp(-\beta\phi(x_1-y_i))-1\right)dy_\Lambda^{\otimes k}\right),$$

see e.g. [Hil56, Eq. (38.16)]. We set

$$\zeta := \max\left\{C_1C_2\exp(\xi I), \xi\right\}.$$

Then (3.5) and (3.6) together with the Ruelle bound (3.3) yield

$$k_{\Lambda_N}^{(n,N)}(x_1,\ldots,x_n) \leq \exp\left(-\sum_{2\leq i\leq n}\beta\phi(x_1-x_j)\right)C_1C_2\left(\xi^{n-1}+\sum_{k=1}^{N-n}\frac{1}{k!}\xi^{n+k-1}I^k\right)$$

$$\leq \exp\left(-\sum_{2\leq i\leq n}\beta\phi(x_1-x_j)\right)\xi^{n-1}C_1C_2\exp(\xi I) \leq \exp\left(-\sum_{2\leq i\leq n}\beta\phi(x_1-x_j)\right)\zeta^n.$$

Finally, symmetry of the correlation functions gives (3.4).   ∎

4. N-PARTICLE STOCHASTIC DYNAMICS IN FINITE VOLUME

Let $\Lambda\subset\mathbb{R}^d$ such that $\Lambda^N\subset\mathbb{R}^{N\cdot d}$ is the closure of an open, relatively compact set, having boundary $\partial(\Lambda^N)$ of Lebesgue measure zero. Our aim is to construct an N-particle diffusion process $(\mathbf{X}(t))_{t\geq 0}$ in $\Gamma_\Lambda^{(N)}$ solving weakly the following N-system of stochastic differential equations before hitting $\partial(\Gamma_\Lambda^{(N)})$:

$$dx(t) = -\beta\sum_{\substack{y(t)\in\mathbf{X}(t)\\y(t)\neq x(t)}}\nabla\phi(x(t)-y(t))\,dt + \sqrt{2}\,dB^{x_0}(t),$$

with reflecting boundary condition,   (4.1)

for sufficiently many initial conditions $\gamma_0 \in \Gamma_\Lambda^{(N)}$. Here $x(t) \in \mathbf{X}(t) \in \Gamma_\Lambda^{(N)}$ and $(B^{x_0})_{x_0 \in \gamma_0}$ are N independent Brownian motions starting in x_0. Existence of a solution to (4.1) was shown in [FG04] by using Dirichlet form techniques. Here we briefly summarize their construction.

First we have to introduce an additional condition:

(D): (*Differentiability*) The function $\exp(-\beta\phi)$ is weakly differentiable on $\mathbb{R}^d$, ϕ is continuously differentiable on $\mathbb{R}^d \backslash \{0\}$ and the gradient $\nabla\phi$, considered as a dx-a.e. defined function on $\mathbb{R}^d$, satisfies

$$\nabla\phi \in L^1(\mathbb{R}^d, \exp(-\beta\phi)dx) \cap L^2(\mathbb{R}^d, \exp(-\beta\phi)dx) \cap L^3(\mathbb{R}^d, \exp(-\beta\phi)dx),$$

where $\beta > 0$ is the inverse temperature. Furthermore, we assume ϕ to be such that the function Φ in (RP) can be chosen differentiable and $\Phi' \exp(-a\Phi)$ a bounded function on $(0, \infty)$ for all $a > 0$.

Note that, for many typical potentials in Statistical Physics, we have $\phi \in C^\infty(\mathbb{R}^d \backslash \{0\})$. For such "outside the origin regular" potentials, condition (D) nevertheless does not exclude a singularity at the point $0 \in \mathbb{R}^d$. The last assumption on ϕ, ensuring a suitable choice of Φ, is no restriction from the physical point of view. E.g., potentials, diverging faster than $\Phi(t) = t^{-d-\epsilon}$, $\epsilon > 0$, at the origin, are admissible.

On Λ^N consider the measure

$$\mu_{\Lambda,N} := \frac{1}{Z_\Lambda^{(N)}} \exp\Big(-\beta \sum_{1 \le i < j \le N} \phi(x_i - x_j)\Big) dx_\Lambda^{\otimes N}.$$

Note that then $\mathrm{sym}^{(N)} : \Lambda^N \to \Gamma_\Lambda^{(N)}$ is $\mu_{\Lambda,N}$-a.e. defined (since the diagonals have $\mu_{\Lambda,N}$-measure zero) and that $\mu_\Lambda^{(N)} = \mu_{\Lambda,N} \circ (\mathrm{sym}^{(N)})^{-1}$. Denote by ∇_i the gradient on $\mathbb{R}^d$ w.r.t. the variable x_i. Then

$$\mathcal{E}_{\Lambda,N}(F, G) := \sum_{1 \le i \le N} \int_{\Lambda^N} (\nabla_i F, \nabla_i G)_{\mathbb{R}^d} \, d\mu_{\Lambda,N}, \quad F, G \in D, \tag{4.2}$$

defines a bilinear from on

$$D = \Big\{ F \in C(\Lambda^N) \,\Big|\, \nabla_i F \text{ locally } dx\text{-integrable on } \mathring{\Lambda}^N, \nabla_i F \in L^2(\mu_{\Lambda,N}) \Big\}. \tag{4.3}$$

Here $(\cdot, \cdot)_{\mathbb{R}^d}$ denotes the scalar product in $\mathbb{R}^d$ inducing the Euclidean norm, $\mathring{A}$ the open kernel of the set A, and the gradient $\nabla_i F$ is meant in the distributional sense on $\mathring{\Lambda}^N$. $(\mathcal{E}_{\Lambda,N}, D)$ is a densely defined, positive definite, symmetric bilinear form on $L^2(\mu_{\Lambda,N})$.

In [FG04, Prop. 5.3] it was shown that $(\mathcal{E}_{\Lambda,N}, D)$ is closable and its closure $(\mathcal{E}_{\Lambda,N}, D(\mathcal{E}_{\Lambda,N}))$ is a conservative, local, quasi-regular Dirichlet form. Thus, there exists a corresponding self-adjoint generator $(H_{\Lambda,N}, D(H_{\Lambda,N}))$ (Friedrichs extension), i.e., $D(H_{\Lambda,N}) \subset D(\mathcal{E}_{\Lambda,N})$ and

$$\mathcal{E}_{\Lambda,N}(F, G) = \int_{\Lambda^N} H_{\Lambda,N} F G \, d\mu_{\Lambda,N}, \qquad F \in D(H_{\Lambda,N}), \, G \in D(\mathcal{E}_{\Lambda,N}).$$

In order to solve (4.1), however, we are rather interested in the image Dirichlet form under $\mathrm{sym}^{(N)}$. Define an isometry $(\mathrm{sym}^{(N)})^* : L^2(\Gamma_\Lambda^{(N)}, \mu_\Lambda^{(N)}) \to L^2(\Lambda^N, \mu_{\Lambda,N})$ by setting

$(\mathrm{sym}^{(N)})^* F$ to be the $\mu_{\Lambda,N}$-class represented by $\tilde{F} \circ \mathrm{sym}^{(N)}$ on $\widetilde{\Lambda^N}$ for any $\mu_\Lambda^{(N)}$-version $\tilde{F}$ of $F \in L^2(\mu_\Lambda^{(N)})$. Note that the subspace

$$L^2_{\mathrm{sym}}(\mu_{\Lambda,N}) := (\mathrm{sym}^{(N)})^*(L^2(\mu_\Lambda^{(N)})) \subset L^2(\mu_{\Lambda,N})$$

is the closed subspace of symmetric functions from $L^2(\mu_{\Lambda,N})$. Using this mapping one can define a bilinear form $(\mathcal{E}_\Lambda^{(N)}, D(\mathcal{E}_\Lambda^{(N)}))$ as the image bilinear form of $(\mathcal{E}_{\Lambda,N}, D(\mathcal{E}_{\Lambda,N}))$ under $\mathrm{sym}^{(N)}$:

$$D(\mathcal{E}_\Lambda^{(N)}) := \{F \in L^2(\mu_\Lambda^{(N)}) \,|\, (\mathrm{sym}^{(N)})^* F \in D(\mathcal{E}_{\Lambda,N})\},$$

$$\mathcal{E}_\Lambda^{(N)}(F,G) := \mathcal{E}_{\Lambda,N}((\mathrm{sym}^{(N)})^* F, (\mathrm{sym}^{(N)})^* G), \qquad F, G \in D(\mathcal{E}_\Lambda^{(N)}). \qquad (4.4)$$

Also $(\mathcal{E}_\Lambda^{(N)}, D(\mathcal{E}_\Lambda^{(N)}))$ is a conservative, local, symmetric Dirichlet form. Its generator is given by

$$H_\Lambda^{(N)} = ((\mathrm{sym}^{(N)})^*)^{-1} \circ H_{\Lambda,N} \circ (\mathrm{sym}^{(N)})^*,$$

$$D(H_\Lambda^{(N)}) = \{F \in L^2(\mu_\Lambda^{(N)}) \,|\, (\mathrm{sym}^{(N)})^* F \in D(H_{\Lambda,N})\}. \qquad (4.5)$$

Of course, $(H_{\Lambda,N}, D(H_{\Lambda,N}))$ generates a strongly continuous contraction semi-group

$$T_\Lambda^{(N)}(t) := \exp(-t H_\Lambda^{(N)}), \quad t \geq 0.$$

For repulsive potentials satisfying (D) in [FG04] it was shown that $Dg := \Lambda^N \setminus \widetilde{\Lambda^N}$ has $\mathcal{E}_{\Lambda,N}$-capacity zero. Thus, $(\mathcal{E}_\Lambda^{(N)}, D(\mathcal{E}_\Lambda^{(N)}))$ is obviously also quasi-regular and by [MR92, Chap. IV, Sect. 3] we have the following theorem:

Theorem 4.1. *Suppose that conditions* (RP), (D) *are satisfied, $N \in \mathbb{N}$ and $\Lambda \subset \mathbb{R}^d$ such that $\Lambda^N \subset \mathbb{R}^{N \cdot d}$ is the closure of an open, relatively compact set with boundary $\partial(\Lambda^N)$ of Lebesgue measure zero. Then:*
(i) There exists a conservative diffusion process (i.e., a conservative strong Markov process with continuous sample paths)

$$\mathbf{M}_\Lambda^{(N)} = \left(\mathbf{\Omega}_\Lambda^{(N)}, \mathbf{F}_\Lambda^{(N)}, (\mathbf{F}_\Lambda^{(N)}(t))_{t \geq 0}, (\mathbf{\Theta}_\Lambda^{(N)}(t))_{t \geq 0}, (\mathbf{X}(t))_{t \geq 0}, (\mathbf{P}_\Lambda^{(N)}(x))_{x \in \Gamma_\Lambda^{(N)}}\right)$$

on $\Gamma_\Lambda^{(N)}$ which is properly associated with $(\mathcal{E}_\Lambda^{(N)}, D(\mathcal{E}_\Lambda^{(N)}))$, i.e., for all ($\mu_\Lambda^{(N)}$-versions of) $F \in L^2(\Gamma_\Lambda^{(N)}, \mu_\Lambda^{(N)})$ and all $t > 0$ the function

$$x \mapsto \int_{\mathbf{\Omega}_\Lambda^{(N)}} F(\mathbf{X}(t)) \, d\mathbf{P}_\Lambda^{(N)}(x), \qquad x \in \Gamma_\Lambda^{(N)},$$

is a $\mathcal{E}_\Lambda^{(N)}$-quasi-continuous version of $T_\Lambda^{(N)}(t) F$. $\mathbf{M}_\Lambda^{(N)}$ is up to $\mu_\Lambda^{(N)}$-equivalence unique. In particular, $\mathbf{M}_\Lambda^{(N)}$ is $\mu_\Lambda^{(N)}$-symmetric (i.e., $\int G T_\Lambda^{(N)}(t) F \, d\mu_\Lambda^{(N)} = \int F T_\Lambda^{(N)}(t) G \, d\mu_\Lambda^{(N)}$ for all $F, G : \Gamma_\Lambda^{(N)} \to [0, \infty)$ measurable) and has $\mu_\Lambda^{(N)}$ as an invariant measure.
(ii) The diffusion process $\mathbf{M}_\Lambda^{(N)}$ is up to $\mu_\Lambda^{(N)}$-equivalence the unique diffusion process having $\mu_\Lambda^{(N)}$ as symmetrizing measure and solving the martingale problem for $(-H_\Lambda^{(N)}, D(H_\Lambda^{(N)}))$, in the sense that for all $G \in D(H_\Lambda^{(N)})$

$$G(\mathbf{X}(t)) - G(\mathbf{X}(0)) + \int_0^t H_\Lambda^{(N)} G(\mathbf{X}(s)) \, ds, \qquad t \geq 0,$$

is an $\mathbf{F}_\Lambda^{(N)}(t)$-martingale under $\mathbf{P}_\Lambda^{(N)}(x)$ (hence starting in x) for $\mathcal{E}_\Lambda^{(N)}$-quasi all $x \in \Gamma_\Lambda^{(N)}$.

In the above theorem $\mathbf{M}_\Lambda^{(N)}$ is canonical, i.e., $\mathbf{\Omega}_\Lambda^{(N)} = C([0,\infty) \to \Gamma_\Lambda^{(N)})$, $\mathbf{X}(t)(\omega) = \omega(t), \omega \in \mathbf{\Omega}_\Lambda^{(N)}$. The filtration $(\mathbf{F}_\Lambda^{(N)}(t))_{t\geq 0}$ is the natural "minimum completed admissible filtration", cf. [FOT94], Chap. A.2, or [MR92], Chap. IV, obtained from the σ-algebras $\sigma\{\omega(s) \,|\, 0 \leq s \leq t, \omega \in \mathbf{\Omega}_\Lambda^{(N)}\}$, $t \geq 0$. $\mathbf{F}_\Lambda^{(N)} := \mathbf{F}_\Lambda^{(N)}(\infty) := \bigvee_{t \in [0,\infty)} \mathbf{F}_\Lambda^{(N)}(t)$ is the smallest σ-algebra containing all $\mathbf{F}_\Lambda^{(N)}(t)$ and $(\mathbf{\Theta}_\Lambda^{(N)}(t))_{t \geq 0}$ are the corresponding natural time shifts. For a detailed discussions of these objects we refer to [MR92].

To illustrate the relation of the process $\mathbf{M}_\Lambda^{(N)}$ to the stochastic differential equation (4.1) we need an explicit representation of the generator $(H_{\Lambda,N}, D(H_{\Lambda,N}))$, at least for some subset of $D(H_{\Lambda,N})$. An integration by parts yields the following representation for $H_{\Lambda,N}$ restricted to $F \in C_c^2(\mathring{\Lambda}^N) \subset D(H_{\Lambda,N})$:

$$H_{\Lambda,N}F(x) = -\sum_{i=1}^N \Delta_i F(x) + \beta \sum_{1 \leq i < j \leq N} \nabla\phi(x_i - x_j)(\nabla_i F(x) - \nabla_j F(x)), \qquad (4.6)$$

$x \in \Lambda^N$. Furthermore, if we assume $\partial(\Lambda^N)$ to be Lipschitz, then

$$\left\{ F \in C^2(\Lambda^N) \,|\, \partial_\nu F = 0 \text{ on } \partial(\Lambda^N) \right\} \subset D(H_{\Lambda,N}), \qquad (4.7)$$

where ∂_ν denotes the normal derivative, and representation (4.6) holds for such functions also, see [FG04, Theo. 3.2]. Note that the functions in (4.7) have Neumann boundary condition on Λ^N.

Let $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ be the set of all functions on Γ of the form

$$F(\gamma) = g_F(\langle f_1, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle), \qquad (4.8)$$

where $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{D} := C_c^\infty(\mathbb{R}^d)$, and $g_F \in C_b^\infty(\mathbb{R}^n)$. Here $C_c^\infty(\mathbb{R}^d)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^d$ with compact support and $C_b^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^n$ which are bounded together with all their derivatives. For F as in (4.8) such that $(\text{sym}^{(N)})^* F \in C_c^2(\mathring{\Lambda}^N)$, (4.6) together with (4.5) yields

$$H_\Lambda^{(N)} F(\gamma) = -\sum_{i,j=1}^N \partial_i \partial_j g_F(\langle f_1, \gamma \rangle, \ldots, \langle f_N, \gamma \rangle)\langle (\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \gamma \rangle$$

$$-\sum_{j=1}^N \partial_j g_F(\langle f_1, \gamma \rangle, \ldots, \langle f_N, \gamma \rangle)\Big(\langle \Delta f_j, \gamma \rangle - \beta \sum_{\{x,y\} \subset \gamma} \nabla\phi(x - y)(\nabla f_j(x) - \nabla f_j(y))\Big),$$

$$(4.9)$$

$\gamma \in \Gamma_\Lambda^{(N)}$, where ∂_j denotes the partial derivative w.r.t. the j-th variable.

Now, using Itô's formula, we find that the process $\mathbf{P}_\Lambda^{(N)}$ solves the stochastic differential equation (4.1) in the sense of the associated martingale problem, see Theorem 4.1(ii). We say that $\mathbf{P}_\Lambda^{(N)}$ corresponds to reflecting boundary condition because of the Neumann boundary condition seen on the level of the domain of its generator, see (4.7).

5. N/V-LIMIT OF N-PARTICLE, FINITE VOLUME STOCHASTIC DYNAMICS

As state space for the N/V-limit we consider $(\Gamma, d_{(\beta/3)\Phi,h})$ with Φ as in condition (RP), (D), and h as in Proposition 2.1. β is the inverse temperature. The laws of the equilibrium processes

$$\mathbf{P}_{\mu_\Lambda^{(N)}} := \int_{\Gamma_\Lambda^{(N)}} \mathbf{P}_\Lambda^{(N)}(\gamma)\, d\mu_\Lambda^{(N)}(\gamma)$$

are probability measures on $C([0,\infty), \Gamma_\Lambda^{(N)})$, cf. Theorem 4.1. Since $C([0,\infty), \Gamma_\Lambda^{(N)})$ is a Borel subset of $C([0,\infty), \Gamma)$ (under the natural embedding) with compatible measurable structures we can consider $\mathbf{P}_{\mu_\Lambda^{(N)}}$ as a measure on $C([0,\infty), \Gamma)$. Below, $(\mathbf{X}(t))_{t\geq 0}$ always denotes the coordinate process in the corresponding path space. We denote by $(\mathbf{F}_t)_{t\geq 0}$ the natural filtration on $C([0,\infty), \Gamma)$.

5.1. **Tightness.**

Theorem 5.1. *Suppose that the conditions* (RP), (T), (BB), *and* (D) *are satisfied and let* $(\Lambda_N)_{N\in\mathbb{N}}$ *be a sequence of volumes corresponding to the density* $\rho > 0$. *Furthermore, assume that the* $\Lambda_N \subset \mathbb{R}^d$ *are such that* $(\Lambda_N)^N \subset \mathbb{R}^{N\cdot d}$ *is the closure of an open, relatively compact set with boundary* $\partial((\Lambda_N)^N)$ *of Lebesgue measure zero. Set* $\mathbf{P}^{(N)} := \mathbf{P}_{\mu_{\Lambda_N}^{(N)}}$. *Then* $(\mathbf{P}^{(N)})_{N\in\mathbb{N}}$ *is tight on* $C([0,\infty), \Gamma)$.

For a symmetric function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ we set functions $f^{[n,2]} : \mathbb{R}^{n\cdot d} \to \mathbb{R}$, $n = 2, 3, 4$, by

$$f^{[2,2]}(x_1, x_2) := f(x_1, x_2)^2,$$

$$f^{[3,2]}(x_1, x_2, x_3) := f(x_1, x_2)f(x_1, x_3) + f(x_1, x_3)f(x_2, x_3) + f(x_2, x_3)f(x_1, x_2),$$

$$f^{[4,2]}(x_1, x_2, x_3, x_4) := f(x_1, x_2)f(x_3, x_4) + f(x_1, x_3)f(x_2, x_4) + f(x_1, x_4)f(x_2, x_3),$$

for $x_1, x_2, x_3, x_4 \in \mathbb{R}$, and functions $f^{[n,3]} : \mathbb{R}^{n\cdot d} \to \mathbb{R}$, $n = 2, \ldots, 6$, by

$$f^{[2,3]}(x_1, x_2) := f(x_1, x_2)^3,$$

$$f^{[3,3]}(x_1, x_2, x_3) := f(x_1, x_2)f(x_1, x_3)f(x_2, x_3) + f(x_1, x_2)^2\big(f(x_1, x_3) + f(x_2, x_3)\big)$$

$$+ f(x_1, x_3)^2\big(f(x_1, x_2) + f(x_2, x_3)\big) + f(x_2, x_3)^2\big(f(x_1, x_2) + f(x_1, x_3)\big),$$

$$f^{[4,3]}(x_1, \ldots, x_4) := f(x_1, x_2)f(x_3, x_4)f(x_1, x_3) + \ldots + f(x_1, x_2)^2 f(x_3, x_4) + \ldots,$$

$$f^{[5,3]}(x_1, \ldots, x_5) := f(x_1, x_2)f(x_3, x_4)f(x_5, x_1) + \ldots,$$

$$f^{[6,3]}(x_1, \ldots, x_6) := f(x_1, x_2)f(x_3, x_4)f(x_5, x_6) + \ldots,$$

for $x_1, \ldots, x_6 \in \mathbb{R}$.

Lemma 5.2. *Let the assumptions in Theorem 5.1 hold and* β, Φ, h *be as in the metric* $d_{(\beta/3)\Phi,h}$. *Set* $\mu^{(N)} := \mu_{\Lambda_N}^{(N)}$. *Then*

$$\sup_{N\in\mathbb{N}} \mathbb{E}_{\mu^{(N)}}\big[(S^{(\beta/3)\Phi,h})^2\big] < \infty.$$

Proof: Set $f(x,y) := \exp((\beta/3)\Phi(|x-y|))h(x)h(y)$, $\{x,y\} \subset \mathbb{R}^d$. Then

$$(S^{(\beta/3)\Phi,h}(\gamma))^2 = \sum_{\{x,y,z,w\}\subset\gamma} f(x,y)f(z,w) + f(x,z)f(y,w) + f(x,w)f(y,z)$$

$$+ \sum_{\{x,y,z\}\subset\gamma} f(x,y)f(y,z) + f(x,z)f(y,z) + f(x,y)f(x,z) + \sum_{\{x,y\}\subset\gamma} f(x,y)^2.$$

Now (3.2) together with (3.4) yields for $N \geq N_0$ (as in Theorem 3.2)

$$\mathbb{E}_{\mu^{(N)}}[(S^{(\beta/3)\Phi,h})^2] = \sum_{n=2}^{4} \frac{1}{n!} \int_{\Lambda_N^n} f^{[n,2]}(x_1, \ldots, x_n) k_{\Lambda_N}^{(n,N)}(x_1, \ldots, x_n) \, dx_{\Lambda_N}^{\otimes n}$$

$$\leq \sum_{n=2}^{4} \frac{\zeta^n}{n!} \int_{(\mathbb{R}^d)^n} |f|^{[n,2]}(x_1, \ldots, x_n) \exp\left(-\frac{2}{n} \sum_{1\leq i<j\leq n} \beta\phi(x_i - x_j)\right) dx^{\otimes n}. \quad (5.1)$$

The integrals in (5.1) are finite due to the integrability properties of h and (RP) (note that $\exp(b\Phi(|\cdot|))\exp(-c\phi)$ is a bounded function for all $c \geq b \geq 0$). Therefore, (5.1) is a bound for $\mathbb{E}_{\mu^{(N)}}[(S^{(\beta/3)\Phi,h})^2]$ uniformly in $N \geq N_0$. Of course, $\mathbb{E}_{\mu^{(N)}}[(S^{(\beta/3)\Phi,h})^2]$ is finite for the finite many $N < N_0$. Thus, the assertion is proven. ∎

Lemma 5.3. Let the assumptions in Theorem 5.1 hold. Then there exists $C_6 < \infty$ such that

$$\sup_{N\in\mathbb{N}} \mathbb{E}_{\mathbf{P}^{(N)}} \left[d_{(\beta/3)\Phi,h}(\mathbf{X}(t), \mathbf{X}(s))^4 \right]^{1/4} \leq C_6 \, (t-s)^{1/2}. \quad (5.2)$$

Proof: Recall the definition of the metric $d_{(\beta/3)\Phi,h}$, see (2.1). Since $|1 - \exp(-r)| \leq r$ for $r \geq 0$, by the triangle inequality we obtain

$$\mathbb{E}_{\mathbf{P}^{(N)}} \left[d_{(\beta/3)\Phi,h}(\mathbf{X}(t), \mathbf{X}(s))^4 \right]^{1/4} \leq \sum_{k=1}^{\infty} 2^{-k} p_k \mathbb{E}_{\mathbf{P}^{(N)}}[|\langle f_k, \mathbf{X}(t)\rangle - \langle f_k, \mathbf{X}(s)\rangle|^4]^{1/4}$$

$$+ \sum_{k=1}^{\infty} 2^{-k} q_k \mathbb{E}_{\mathbf{P}^{(N)}}[|S^{(\beta/3)\Phi,h_k}(\mathbf{X}(t)) - S^{(\beta/3)\Phi,h_k}(\mathbf{X}(s))|^4]^{1/4}. \quad (5.3)$$

Set $F(x) := \sum_{1\leq i\leq N} f(x_i)$, $x \in (\Lambda_N)^N$, $f \in C_c^1(\mathbb{R}^d)$. By (4.3) we know that $F \in D(\mathcal{E}_{\Lambda_N,N})$. Note that $\langle f, \mathrm{sym}^{(N)}(\cdot)\rangle = F$ on $\widetilde{(\Lambda_N)^N}$. Thus, by (4.4) $\langle f, \cdot\rangle \in D(\mathcal{E}_{\Lambda_N}^{(N)})$. Fix $T > 0$. Below we canonically project the laws of the equilibrium processes $\mathbf{P}^{(N)}$ onto $\Omega_T^{(N)} := C([0,T], \Gamma_{\Lambda_N}^{(N)})$ without expressing this explicitly. We define the time reversal $r_T(\omega) := \omega(T - \cdot)$, $\omega \in \Omega_T^{(N)}$. Now, by the well-known Lyons–Zheng decomposition, cf. [LZ88], [FOT94], we have for all $0 \leq t \leq T$:

$$\langle f, \mathbf{X}(t)\rangle - \langle f, \mathbf{X}(0)\rangle = \frac{1}{2}\mathbf{M}(N,f,t) + \frac{1}{2}\Big(\mathbf{M}(N,f,T-t)(r_T) - \mathbf{M}(N,f,T)(r_T)\Big)$$

$\mathbf{P}^{(N)}$-a.e., where $(\mathbf{M}(N,f,t))_{0\leq t\leq T}$ is a continuous $(\mathbf{P}^{(N)}, (\mathbf{F}_{\Lambda_N}^{(N)}(t))_{0\leq t\leq T})$-martingale and $(\mathbf{M}(N,f,t)(r_T))_{0\leq t\leq T}$ is a continuous $(\mathbf{P}^{(N)}, (r_T^{-1}(\mathbf{F}_{\Lambda_N}^{(N)}(t)))_{0\leq t\leq T})$-martingale. (We note

that $\mathbf{P}^{(N)} \circ r_T^{-1} = \mathbf{P}^{(N)}$ because $(T_{\Lambda_N}^{(N)}(t))_{t \geq 0}$ is symmetric on $L^2(\mu^{(N)})$.) Moreover, by (4.2) the bracket of $\mathbf{M}(N, f)$ is given by

$$< \mathbf{M}(N, f) > (t) = \int_0^t \langle |\nabla f|_{\mathbb{R}^d}^2, \mathbf{X}(u) \rangle \, du$$

as e.g. directly follows from [FOT94], Theorem 5.2.3 and Theorem 5.1.3(i). Hence by the Burkholder–Davies–Gundy inequalities and since $\mathbf{P}_N \circ r_T^{-1} = \mathbf{P}_N$ we can find $C_7 \in (0, \infty)$ such that for all $f \in C_c^1(\mathbb{R}^d)$, $N \geq N_0$ (as in Theorem 3.2), $0 \leq s \leq t \leq T$,

$$\mathbb{E}_{\mathbf{P}^{(N)}}[|\langle f, \mathbf{X}(t) \rangle - \langle f, \mathbf{X}(s) \rangle|^4]^{1/4} \leq \frac{1}{2} \Big(\mathbb{E}_{\mathbf{P}^{(N)}}[|\mathbf{M}(N, f, t) - \mathbf{M}(N, f, s)|^4]^{1/4}$$

$$+ \mathbb{E}_{\mathbf{P}^{(N)}}[|\mathbf{M}(N, f, T - t)(r_T) - \mathbf{M}(N, f, T - s)(r_T)|^4]^{1/4} \Big)$$

$$\leq C_7 \left(\mathbb{E}_{\mathbf{P}^{(N)}} \Big[\Big(\int_s^t \langle |\nabla f|_{\mathbb{R}^d}^2, \mathbf{X}(u) \rangle \, du \Big)^2 \Big]^{1/4} \right.$$

$$+ \mathbb{E}_{\mathbf{P}^{(N)}} \Big[\Big(\int_{T-t}^{T-s} \langle |\nabla f|_{\mathbb{R}^d}^2, \mathbf{X}(T - u) \rangle \, du \Big)^2 \Big]^{1/4} \right)$$

$$\leq 2 \, C_7 \, (t - s)^{1/2} \Big(\int_{\Gamma_{\Lambda_N}^{(N)}} \langle |\nabla f|_{\mathbb{R}^d}^2, \gamma \rangle^2 \, d\mu^{(N)}(\gamma) \Big)^{1/4}$$

$$\leq C_8 \, (t - s)^{1/2} \Big(\Big(\int_{\Lambda_N} |\nabla f|^2 \, dx_{\Lambda_N} \Big)^2 + \Big(\int_{\Lambda_N} |\nabla f|^4 \, dx_{\Lambda_N} \Big) \Big)^{1/4} \leq C_8 \, (t - s)^{1/2} I(f),$$

$$(5.4)$$

where $C_8 := 2 \, C_7 \max\{\xi^2/2, \xi\}^{1/4}$, see (3.2) together with Theorem 3.2, and $I(f)$ is given by

$$I(f) = \Big(\Big(\int_{\mathbb{R}^d} |\nabla f|^2 \, dx \Big)^2 + \Big(\int_{\mathbb{R}^d} |\nabla f|^4 \, dx \Big) \Big)^{1/4}.$$

But then from the above derivation of (5.4) it is clear there exists $C_9 < \infty$ such that

$$\mathbb{E}_{\mathbf{P}^{(N)}}[|\langle f, \mathbf{X}(t) \rangle - \langle f, \mathbf{X}(s) \rangle|^4]^{1/4} \leq C_9 I(f) \, (t - s)^{1/2} \qquad (5.5)$$

for all $f \in C_c^1(\mathbb{R}^d)$, $N \in \mathbb{N}$, $0 \leq s \leq t \leq T$.

Now set $U(x) = \sum_{1 \leq i < j \leq N} \exp((\beta/3)\Phi(|x_i - x_j|)) f(x_i) f(x_j)$, $x \in \widetilde{(\Lambda_N)^N}$, $f \in C_c^1(\mathbb{R}^d)$, non-negative, and Φ as in condition (RP), (D). Then from (4.3) together with an approximation argument we can conclude that $U \in D(\mathcal{E}_{\Lambda_N, N})$. This together with the fact that $S^{(\beta/3)\Phi, f}(\mathrm{sym}^{(N)}(\cdot)) = U$ on $\widetilde{(\Lambda_N)^N}$ implies via (4.4) that $S^{(\beta/3)\Phi, f} \in D(\mathcal{E}_{\Lambda_N}^{(N)})$. Hence as above we can find a $C_{10} < \infty$ such that for all non-negative $f \in C_c^1(\mathbb{R}^d)$, $N \in \mathbb{N}$, $0 \leq s \leq t \leq T$,

$$\mathbb{E}_{\mathbf{P}^{(N)}}[|S^{(\beta/3)\Phi, f}(\mathbf{X}(t)) - S^{(\beta/3)\Phi, f}(\mathbf{X}(s))|^4]^{1/4} \leq C_{10} \, (t-s)^{1/2} \Big(\int_{\Gamma_{\Lambda_N}^{(N)}} G(\gamma) \, d\mu^{(N)}(\gamma) \Big)^{1/4},$$

$$(5.6)$$

where

$$G(\gamma) = \sum_{x \in \gamma} \sum_{y \in \gamma \setminus \{x\}} \sum_{z \in \gamma \setminus \{x\}} \exp((\beta/3)\Phi(|x-y|)) \exp((\beta/3)\Phi(|x-z|)) f(y) f(z)$$

$$\times \left(\frac{\beta \Phi'(|x-y|)}{3|x-y|} \frac{\beta \Phi'(|x-z|)}{3|x-z|} (x-y, x-z)_{\mathbb{R}^d} f(x)^2 + |\nabla f(x)|_{\mathbb{R}^d}^2 \right.$$

$$\left. + \frac{\beta \Phi'(|x-y|)}{3|x-y|} (x-y, \nabla f(x))_{\mathbb{R}^d} f(x) + \frac{\beta \Phi'(|x-z|)}{3|x-z|} (x-z, \nabla f(x))_{\mathbb{R}^d} f(x) \right)$$

$$= \sum_{\{x,y\} \in \gamma} g_2^f(x,y) + \sum_{\{x,y,z\} \in \gamma} g_3^f(x,y,z).$$

The function g_2^f is given by

$$g_2^f(x,y) = \exp((2/3)\beta \Phi(|x-y|)) \left(\frac{2\beta^2}{9} \Phi'(|x-y|)^2 f(x)^2 f(y)^2 + |\nabla f(x)|_{\mathbb{R}^d}^2 f(y)^2 \right.$$

$$\left. + |\nabla f(y)|_{\mathbb{R}^d}^2 f(x)^2 + \frac{2\beta \Phi'(|x-y|)}{3|x-y|} (x-y, f(y)^2 f(x)\nabla f(x) - f(x)^2 f(y)\nabla f(y))_{\mathbb{R}^d} \right)$$

and g_3^f is the symmetrization of

$$6 \exp((\beta/3)\Phi(|x-y|)) \exp((\beta/3)\Phi(|x-z|)) f(y) f(z)$$

$$\times \left(\frac{\beta \Phi'(|x-y|)}{3|x-y|} \frac{\beta \Phi'(|x-z|)}{3|x-z|} (x-y, x-z)_{\mathbb{R}^d} f(x)^2 \right.$$

$$+ |\nabla f(x)|_{\mathbb{R}^d}^2 + \frac{\beta \Phi'(|x-y|)}{3|x-y|} (x-y, \nabla f(x))_{\mathbb{R}^d} f(x) + \frac{\beta \Phi'(|x-z|)}{3|x-z|} (x-z, \nabla f(x))_{\mathbb{R}^d} f(x) \right).$$

Now by (3.2) together with Theorem 3.2 we get for all non-negative $f \in C_c^1(\mathbb{R}^d)$, $N \geq N_0$, $0 \leq s \leq t \leq T$, the following estimate:

$$\mathbb{E}_{\mathbf{P}^{(N)}}[|S^{(\beta/3)\Phi,f}(\mathbf{X}(t)) - S^{(\beta/3)\Phi,f}(\mathbf{X}(s))|^4]^{1/4} \leq C_{10}(t-s)^{1/2} R(f), \tag{5.7}$$

where

$$R(f) := \left(\frac{\zeta^3}{3!} \int_{(\mathbb{R}^d)^3} |g_3^f(x_1, x_2, x_3)| \exp\left(-\frac{2}{3} \sum_{1 \leq i < j \leq 3} \beta \phi(x_i - x_j) \right) dx^{\otimes 3} \right.$$

$$\left. + \frac{\zeta^2}{2!} \int_{(\mathbb{R}^d)^2} |g_2^f(x_1, x_2)| \exp\left(-\beta \phi(x_1 - x_2) \right) dx^{\otimes 2} \right)^{1/4}. \tag{5.8}$$

The integrals in (5.8) are finite due to the differentiability and integrability properties of f and (RP), (D) ($\Phi' \exp(-a\Phi)$ is by assumption a bounded function for all $a > 0$). Then for all non-negative $f \in C_c^1(\mathbb{R}^d)$, $N \geq N_0$, $0 \leq s \leq t \leq T$,

$$\mathbb{E}_{\mathbf{P}^{(N)}}[|S^{(\beta/3)\Phi,f}(\mathbf{X}(t)) - S^{(\beta/3)\Phi,f}(\mathbf{X}(s))|^4]^{1/4} \leq C_{10} R(f)(t-s)^{1/2}.$$

But then by (5.6) there exists $C_{11} < \infty$ such that

$$\mathbb{E}_{\mathbf{P}^{(N)}}[|S^{(\beta/3)\Phi,f}(\mathbf{X}(t)) - S^{(\beta/3)\Phi,f}(\mathbf{X}(s))|^4]^{1/4} \leq C_{11} R(f)(t-s)^{1/2} \tag{5.9}$$

for all non-negative $f \in C_c^1(\mathbb{R}^d)$, $N \in \mathbb{N}$, $0 \leq s \leq t \leq T$. If we now assume that

$$q_k = \inf\{1, 1/I(f_k)\} > 0 \quad \text{and} \quad p_k = \inf\{1, 1/R(h_k)\} > 0,$$

and $C_6 = C_9 + C_{11}$, then from (5.3) together with (5.5) and (5.9) we can conclude (5.2). ∎

Proof of Theorem 5.1: Criteria for tightness of càdlàg (i.e., right continuous on $[0, \infty)$ and left limits on $(0, \infty)$) processes in metric spaces have been worked out in [EK86, Chap. 3]. For continuous processes as we are considering one uses a slightly different modulus of continuity (and also a different topology on the path space) as for càdlàg processes. However, by using the Arzela–Ascoli Theorem in metric spaces, see e.g. [Cho66, Chap. I, Theo. 23.2], it is easy to show that [EK86, Chap. 3, Theo. 7.2] is also valid in the continuous case. Since the sets

$$\{\gamma \in \Gamma \mid S_{\Phi,h}(\gamma) \leq R\}, \qquad R < \infty,$$

are relatively compact subsets of $(\Gamma, d_{(\beta/3)\Phi,h})$, see Proposition 2.1, Lemma 5.2 yields condition (a) of [EK86, Chap. 3, Theo. 7.2] (recall that $\mu^{(N)}$ is the invariant measure of $\mathbf{P}^{(N)}$). Condition (b) of [EK86, Chap. 3, Theo. 7.2] follows from Lemma 5.3. ∎

5.2. Identification of the limiting equilibrium measures as a canonical Gibbs measures.

Consider the sequence of equilibrium measures $(\mu^{(N)})_{N \in \mathbb{N}}$ corresponding to the $(\mathbf{P}^{(N)})_{N \in \mathbb{N}}$ as in Theorem 5.1. Then tightness of $(\mathbf{P}^{(N)})_{N \in \mathbb{N}}$ implies tightness of $(\mu^{(N)})_{N \in \mathbb{N}}$. Now let μ be an accumulation point of $(\mu^{(N)})_{N \in \mathbb{N}}$. Our aim is to identify μ as a canonical Gibbs measure via an integration by parts formula.

Lemma 5.4. Assume condition (D). For $n \in \mathbb{N}$ and $v \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ consider the function

$$\Gamma \ni \gamma \mapsto L_{v,k}^\phi(\gamma) := -\beta \sum_{\{x,y\} \in \gamma} (\nabla \phi(x-y), I_k(y)v(x) - I_k(x)v(y))_{\mathbb{R}^d},$$

where the collection $\mathbf{I} = \{I_k \mid k \in \mathbb{N}\}$ is as in Section 2. Then $L_{v,k}^\phi$ is a continuous function on $(\Gamma, d_{(\beta/3)\Phi,h})$.

Proof: Just an easy modification of the proof of [KK04, Lem. 3.4] where the continuity of $S^{(\beta/3)\Phi,h_k}$ is shown. ∎

Lemma 5.5. Let the conditions in Theorem 5.1 hold. Then for all accumulation points μ of $(\mu^{(N)})_{N \in \mathbb{N}}$ and all $v \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ we have that

$$L_v^{\phi,\mu} := \lim_{k \to \infty} L_{v,k}^\phi$$

exists in $L^2(\mu)$.

Proof: Set $f_k(x,y) := \beta \nabla \phi(x-y))\Big(I_k(y)v(x) - I_k(x)v(y)\Big)$, $\{x,y\} \subset \mathbb{R}^d$. Then

$$(L_{v,k}^\phi(\gamma))^2 = \sum_{n=2}^4 \sum_{\{x_1,\dots,x_n\} \subset \gamma} f_k^{[n,2]}(x_1,\dots,x_n).$$

Now as in the proof of Lemma 5.2, (3.2) together with (3.4) yields for $N \geq N_0$ (as in Theorem 3.2)

$$\mathbb{E}_{\mu^{(N)}}[(L_{v,k}^\phi)^2] \leq J_2 + J_3 + J_4,$$

where

$$J_n = \frac{\zeta^n}{n!} \int_{(\mathbb{R}^d)^n} |f_k|^{[n,2]}(x_1, \ldots, x_n) \exp\left(-\frac{2}{n} \sum_{1 \leq i < j \leq n} \beta \phi(x_i - x_j)\right) dx^{\otimes n}, \quad n = 2, 3, 4.$$

Since the potential ϕ is bounded from below, there exits $C_{12} < \infty$ such that

$$|J_4| \leq \frac{C_{12}}{4} \Big(\int_{(\mathbb{R}^d)^2} \|I_k(x_2)v(x_1) - I_k(x_1)v(x_2)\|_{\mathbb{R}^d}$$

$$\times \|\beta \nabla\phi(x_1 - x_2)\|_{\mathbb{R}^d} \exp\left(-\beta\phi(x_1 - x_2)\right) dx^{\otimes 2} \Big)^2 \leq C_{12} \|v\|_{L^1(dx)}^2 \|\beta \nabla\phi\|_{L^1(\exp(-\beta\phi)dx)}^2.$$

Analogously, (using Young's inequality) we obtain

$$|J_3| \leq \frac{C_{13}}{2} \|v\|_{\sup} \|\beta \nabla\phi\|_{L^1(\exp(-\beta\phi)dx)}$$

$$\times \int_{(\mathbb{R}^d)^2} \|I_k(x_2)v(x_1) - I_k(x_1)v(x_2)\|_{\mathbb{R}^d} \|\beta \nabla\phi(x_1 - x_2)\|_{\mathbb{R}^d} \exp\left(-\beta\phi(x_1 - x_2)\right) dx^{\otimes 2}$$

$$\leq C_{13} \|v\|_{\sup} \|v\|_{L^1(dx)} \|\beta \nabla\phi\|_{L^1(\exp(-\beta\phi)dx)}^2$$

for some $C_{13} < \infty$. J_2 can be estimated as follows:

$$|J_2| \leq \frac{C_{14}}{4} \int_{(\mathbb{R}^d)^2} \|I_k(x_2)v(x_1) - I_k(x_1)v(x_2)\|_{\mathbb{R}^d}^2$$

$$\times \|\beta \nabla\phi(x_1 - x_2)\|_{\mathbb{R}^d}^2 \exp\left(-\beta\phi(x_1 - x_2)\right) dx^{\otimes 2} \leq C_{14} \|v\|_{L^2(dx)}^2 \|\beta \nabla\phi\|_{L^2(\exp(-\beta\phi)dx)}^2.$$

Next for $0 < r < \infty$ set

$$L_{v,k,r}^\phi := (L_{v,k,r}^\phi \vee -r) \wedge r.$$

Then, by Lemma 5.4, $L_{v,k,r}^\phi$ is a bounded continuous function on $(\Gamma, d_{(\beta/3)\Phi,h})$. Additionally,

$$(L_{v,k,r}^\phi)^2 \leq (L_{v,k}^\phi)^2 \quad \text{and} \quad (L_{v,k,r}^\phi)^2 \nearrow (L_{v,k}^\phi)^2 \quad \text{as} \quad r \nearrow \infty.$$

Now let μ be an accumulation point of $(\mu^{(N)})_{N \in \mathbb{N}}$, i.e., $\mu^{(N_n)} \to \mu$ weakly for some subsequence $N_n \to \infty$ as $n \to \infty$. Then

$$\mathbb{E}_\mu[(L_{v,k,r}^\phi)^2] = \lim_{n \to \infty} \mathbb{E}_{\mu^{(N_n)}}[(L_{v,k,r}^\phi)^2] \leq \liminf_{n \to \infty} \mathbb{E}_{\mu^{(N_n)}}[(L_{v,k}^\phi)^2]$$

and so

$$\mathbb{E}_\mu[(L_{v,k}^\phi)^2] = \lim_{r \to \infty} \mathbb{E}_\mu[(L_{v,k,r}^\phi)^2] \leq \liminf_{n \to \infty} \mathbb{E}_{\mu^{(N_n)}}[(L_{v,k}^\phi)^2] < \infty$$

due to the estimates for $|J_2|, |J_3|, |J_4|$. Hence $L_{v,k}^\phi \in L^2(\mu)$. Let $k \geq l$. As above we can estimate

$$\mathbb{E}_\mu[(L_{v,k}^\phi - L_{v,l}^\phi)^2] \leq \liminf_{n\to\infty} \mathbb{E}_{\mu^{(N_n)}}[(L_{v,k}^\phi - L_{v,l}^\phi)^2] \leq C_{15}\left(\|v\|_{\sup}\|\beta\,\nabla\phi\|_{L^1(\exp(-\beta\phi)dx)}\right.$$

$$\times \int_{(\mathbb{R}^d)^2} \|(I_k(x_2) - I_l(x_2))v(x_1) - (I_k(x_1) - I_l(x_1))v(x_2)\|_{\mathbb{R}^d}$$

$$\times \|\beta\,\nabla\phi(x_1 - x_2)\|_{\mathbb{R}^d} \exp\left(-\beta\phi(x_1 - x_2)\right) dx^{\otimes 2}$$

$$+ \int_{(\mathbb{R}^d)^2} \|(I_k(x_2) - I_l(x_2))v(x_1) - (I_k(x_1) - I_l(x_1))v(x_2)\|_{\mathbb{R}^d}^2$$

$$\left. \times \|\beta\,\nabla\phi(x_1 - x_2)\|_{\mathbb{R}^d}^2 \exp\left(-\beta\phi(x_1 - x_2)\right) dx^{\otimes 2}\right)$$

$$\leq C_{15}\left(2\|v\|_{\sup}\|v\|_{L^1(dx)}\|\beta\,\nabla\phi\|_{L^1(\exp(-\beta\phi)dx)}\right.$$

$$\times \int_{\mathbb{R}^d\backslash B_{R_l}(0)} \|\beta\,\nabla\phi(x_1 - x_2)\|_{\mathbb{R}^d} \exp\left(-\beta\phi(x_1 - x_2)\right) dx$$

$$+ 2\|v\|_{L^2(dx)}^2 \int_{\mathbb{R}^d\backslash B_{R_l}(0)} \|\beta\,\nabla\phi(x_1 - x_2)\|_{\mathbb{R}^d}^2 \exp\left(-\beta\phi(x_1 - x_2)\right) dx\right) := C_{16}(l), \quad (5.10)$$

where R_l is a certain radius and $B_{R_l}(0)$ the corresponding ball centered at the origin. Since $I_k(x) - I_l(x) = 0$ if $\|x\| \leq l - 1$, we have $R_l \to \infty$ as $l \to \infty$. Now property (D) yields that $C_{16}(l) \to 0$ as $l \to \infty$. Hence $(L_{v,k}^\phi)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mu)$. ∎

Lemma 5.6. Let the conditions in Theorem 5.1 hold, let $v \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and define

$$L_v^\phi(\gamma) := \begin{cases} -\beta \sum_{\{x,y\}\in\gamma}(\nabla\phi(x-y), v(x) - v(y))_{\mathbb{R}^d} \\ \qquad\qquad \text{if} \sum_{\{x,y\}\in\gamma} |(\nabla\phi(x-y), v(x))_{\mathbb{R}^d}| < \infty \\ \qquad\qquad 0 \qquad\qquad\qquad \text{otherwise} \end{cases}.$$

Then L_v^ϕ is an $L^2(\mu)$-version of $L_v^{\phi,\mu}$.

Proof: Define the sequence

$$M_{v,k}^\phi(\gamma) = \sum_{\{x,y\}\in\gamma} |(\nabla\phi(x-y), v(x))_{\mathbb{R}^d}| I_k(y), \quad k \in \mathbb{N}.$$

Then $M_{v,k}^\phi(\gamma)$ monotonically converges to

$$M_v^\phi(\gamma) := \sum_{\{x,y\}\in\gamma} |(\nabla\phi(x-y), v(x))_{\mathbb{R}^d}|$$

as $k \to \infty$. Furthermore, by estimates as in the proof of Lemma 5.5, the $L^1(\mu)$-norms of the $M_{v,k}^\phi(\gamma)$ are uniformly bounded. Thus, by monotone convergence $M_v^\phi \in L^1(\mu)$ and therefore there exists $S \subset \Gamma$ with $\mu(S) = 1$ such that

$$M_v^\phi(\gamma) < \infty \quad \text{for all} \quad \gamma \in S.$$

Now we return to the $L_{v,k}^{\phi}$. Note that for a subsequence $L_{v,k_m}^{\phi}(\gamma) \to L_v^{\phi,\mu}(\gamma)$ as $m \to \infty$ for μ-a.a. $\gamma \in \Gamma$. Obviously, for this subsequence and all $\gamma \in S$: $L_{v,k_m}^{\phi}(\gamma) \to L_v^{\phi}(\gamma)$ as $m \to \infty$. Thus

$$L_v^{\phi,\mu}(\gamma) = \lim_{m \to \infty} L_{v,k_m}^{\phi}(\gamma) = L_v^{\phi}(\gamma) \quad \text{for } \mu \text{ a.a.} \quad \gamma \in S.$$

∎

For later use we also need:

Lemma 5.7. Let the conditions in Theorem 5.1 hold. Then for all subsequences $(\mu^{(N_n)})_{n \in \mathbb{N}}$ converging weakly to an accumulation point μ of $(\mu^{(N)})_{N \in \mathbb{N}}$ and all $v \in C_c^{\infty}(\mathbb{R}^d, \mathbb{R}^d)$ we have

$$\sup_{k \in \mathbb{N}} \mathbb{E}_{\mu}[|L_{v,k}^{\phi}|^3] \leq \sup_{k \in \mathbb{N}} \sup_{n \in \mathbb{N}} \mathbb{E}_{\mu^{(N_n)}}[|L_{v,k}^{\phi}|^3] < \infty.$$

Proof: As in the proof of Lemma 5.5 we get for $N \geq N_0$ (as in Theorem 3.2):

$$\mathbb{E}_{\mu^{(N)}}[|L_{v,k}^{\phi}|^3] \leq K_2 + K_3 + K_4 + K_5 + K_6,$$

where

$$K_n = \frac{\zeta^n}{n!} \int_{(\mathbb{R}^d)^n} |f_k|^{[n,3]}(x_1, \dots, x_n) \exp\left(-\frac{2}{n} \sum_{1 \leq i < j \leq n} \beta\phi(x_i - x_j)\right) dx^{\otimes n}, \quad n = 2, \dots, 6.$$

K_2 can be estimated as J_2, here we need that $\nabla\phi \in L^3(\mathbb{R}^d, \exp(-\beta\phi)dx)$. K_6 can be estimated as J_4, here we need that $\nabla\phi \in L^1(\mathbb{R}^d, \exp(-\beta\phi)dx)$. Using Young's inequality $K3$-$K5$ can be treated as $J3$. In these cases we need $\nabla\phi \in L^1(\mathbb{R}^d, \exp(-\beta\phi)dx) \cap L^2(\mathbb{R}^d, \exp(-\beta\phi)dx)$. Then as in the proof of Lemma 5.5 we get the desired estimate. ∎

In order to formulate the next lemma we recall the gradient ∇^{Γ} introduced and studied in [AKR98a]. It acts on finitely based smooth functions as in (4.8) as follows:

$$(\nabla^{\Gamma} F)(\gamma, x) = \sum_{j=1}^{n} \partial_j g_F(\langle f_1, \gamma \rangle, \dots, \langle f_n, \gamma \rangle) \nabla f_j(x), \quad \gamma \in \Gamma, \, x \in \gamma.$$

The corresponding directional derivative ∇_v^{Γ} in direction $v \in C_c^{\infty}(\mathbb{R}^d, \mathbb{R}^d)$ is given by

$$(\nabla_v^{\Gamma} F)(\gamma) = \sum_{j=1}^{n} \partial_j g_F(\langle f_1, \gamma \rangle, \dots, \langle f_n, \gamma \rangle)\langle (\nabla f_j, v)_{\mathbb{R}^d}, \gamma \rangle \quad \gamma \in \Gamma, \, x \in \gamma. \qquad (5.11)$$

Lemma 5.8. Suppose that the conditions (BB), (D) are satisfied and that $N \in \mathbb{N}$, $\Lambda \subset \mathbb{R}^d$ bounded Borel measurable. Let $\mu_{\Lambda}^{(N)}$ be the corresponding canonical Gibbs measure. Then for all $F, G \in \mathcal{F}C_b^{\infty}(\mathcal{D}, \Gamma)$ and $v \in C_c^{\infty}(\mathring{\Lambda}, \mathbb{R}^d)$ the following integration by parts formula holds:

$$\int_{\Gamma_{\Lambda}^{(N)}} \nabla_v^{\Gamma} F G \, d\mu_{\Lambda}^{(N)} = -\int_{\Gamma_{\Lambda}^{(N)}} F \nabla_v^{\Gamma} G \, d\mu_{\Lambda}^{(N)} - \int_{\Gamma_{\Lambda}^{(N)}} FG B_v^{\phi} \, d\mu_{\Lambda}^{(N)}, \qquad (5.12)$$

where

$$B_v^{\phi} := \langle \operatorname{div} v, \cdot \rangle + L_v^{\phi}. \qquad (5.13)$$

Proof: First let us show that $\langle \operatorname{div} v, \cdot \rangle \in L^2(d\mu_\Lambda^{(N)})$. Indeed, by (3.2)

$$\int_{\Gamma_\Lambda^{(N)}} \langle \operatorname{div} v, \gamma \rangle^2 \, d\mu_\Lambda^{(N)}(\gamma) = \int_{\Gamma_\Lambda^{(N)}} \sum_{\{x,y\} \in \gamma} \operatorname{div} v(x) \operatorname{div} v(y) + \sum_{x \in \gamma} (\operatorname{div} v(x))^2 \, d\mu_\Lambda^{(N)}(\gamma)$$

$$= \int_\Lambda (\operatorname{div} v(x))^2 k_\Lambda^{(1,N)}(x) \, dx_\Lambda + \frac{1}{2} \int_{\Lambda^2} \operatorname{div} v(x_1) \operatorname{div} v(x_2) \, k_\Lambda^{(2,N)}(x_1, x_2) \, dx_\Lambda^{\otimes 2} \quad (5.14)$$

which is finite due to the boundedness of the correlation functions. Similarly, we find that $\langle (\nabla f, v)_{\mathbb{R}^d}, \cdot \rangle \in L^2(d\mu_\Lambda^{(N)})$ for all $f \in \mathcal{D}$.

Next note that

$$B_v^\phi(\gamma) = \langle \operatorname{div} v, \gamma \rangle \; + L_{v,k}^\phi(\gamma)$$

for all $\gamma \in \Gamma_\Lambda^{(N)}$ if k is chosen large enough. Then using the ideas as in the proof of Lemma 5.5 one gets $L_{v,k}^\phi \in L^2(\mu_\Lambda^{(N)})$ and then, of course, also $B_v^\phi \in L^2(\mu_\Lambda^{(N)})$. Now by going to Euclidean coordinates one easily proves (5.12) by integrating by parts. Note that the boundary terms are zero due to the support property of v and that (D) implies that

$$\nabla \exp(-\phi) = -\nabla \phi \exp(-\phi) \quad dx\text{-a.e. on } \mathbb{R}^d.$$

∎

Theorem 5.9. *Assume the conditions in Theorem 5.1. Furthermore, let μ be an accumulation point of $(\mu^{(N)})_{N \in \mathbb{N}}$ provided by Theorem 5.1. Then for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ and $v \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ the following integration by parts formula holds:*

$$\int_\Gamma \nabla_v^\Gamma F G \, d\mu = -\int_\Gamma F \nabla_v^\Gamma G \, d\mu - \int_\Gamma F G B_v^\phi \, d\mu. \tag{5.15}$$

In particular, μ is a canonical Gibbs measure.

Proof: By the product rule for ∇^Γ it suffices to prove (5.15) for

$$F = g_F(\langle f_1, \cdot \rangle, \ldots, \langle f_n, \cdot \rangle) \quad \text{and} \quad G \equiv 1.$$

If now $\mu^{(N_n)} \to \mu$ weakly as $n \to \infty$, then by Lemma 5.8 it suffices to show that

$$\lim_{n \to \infty} \mathbb{E}_{\mu^{(N_n)}}[\nabla_v^\Gamma F] = \mathbb{E}_\mu[\nabla_v^\Gamma F] \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}_{\mu^{(N_n)}}[F B_v^\phi] = \mathbb{E}_\mu[F B_v^\phi]. \tag{5.16}$$

Let us first consider the second identity in (5.16). From (5.14) together with (3.3) we can conclude that

$$\mathbb{E}_{\mu^{(N)}}[\langle \operatorname{div} v, \cdot \rangle^2] \le \xi \|\operatorname{div} v\|_{L^2(dx)}^2 + \frac{\xi^2}{2} \|\operatorname{div} v\|_{L^1(dx)}^2 := C_{17}. \tag{5.17}$$

Furthermore, notice that for $0 < r < \infty$

$$D_{v,r} := (\langle \operatorname{div} v, \cdot \rangle \vee -r) \wedge r$$

is a bounded continuous function on $(\Gamma, d_{(\beta/3)\Phi, h})$. Hence

$$\mathbb{E}_\mu[D_{v,r}^2] \le \mathbb{E}_\mu[\langle \operatorname{div} v, \cdot \rangle^2] \le \xi \|\operatorname{div} v\|_{L^2(dx)}^2 + \frac{\xi^2}{2} \|\operatorname{div} v\|_{L^1(dx)}^2,$$

by the same arguments as in the proof of Lemma 5.5 (there applied to $L_{v,k}^\phi$). Then by the triangle inequality, (5.10) and (5.17)

$$
\begin{aligned}
|\mathbb{E}_{\mu^{(N_n)}}[FB_v^\phi] - \mathbb{E}_\mu[FB_v^\phi]| &\leq \mathbb{E}_{\mu^{(N_n)}}[|F||\langle \operatorname{div} v, \cdot \rangle - D_{v,r}|] \\
&+ |\mathbb{E}_{\mu^{(N_n)}}[FD_{v,r}] - \mathbb{E}_\mu[FD_{v,r}]| + \mathbb{E}_\mu[|F||\langle \operatorname{div} v, \cdot \rangle - D_{v,r}|] \\
&+ \mathbb{E}_{\mu^{(N_n)}}\Big[|F|\Big|\sum_{\{x,y\}\in(\cdot)} (\nabla\phi(x-y), (1-I_k(y))v(x) - (1-I_k(x))v(y))_{\mathbb{R}^d}\Big|\Big] \\
&+ \mathbb{E}_{\mu^{(N_n)}}[|F||L_{v,k}^\phi - L_{v,k,r}^\phi|] + |\mathbb{E}_{\mu^{(N_n)}}[FL_{v,k,r}^\phi] - \mathbb{E}_\mu[FL_{v,k,r}^\phi]| \\
&+ \mathbb{E}_\mu[|F||L_{v,k,r}^\phi - L_{v,k}^\phi|] + \mathbb{E}_\mu[|F||L_{v,k}^\phi - L_v^\phi|] \\
&\leq \frac{2C_{17}\|g_F\|_{\sup}}{r} + |\mathbb{E}_{\mu^{(N_n)}}[FD_{v,r}] - \mathbb{E}_\mu[FD_{v,r}]| \\
&+ \|g_F\|_{\sup}C_{16}(k) + |\mathbb{E}_{\mu^{(N_n)}}[FL_{v,k,r}^\phi] - \mathbb{E}_\mu[FL_{v,k,r}^\phi]| + \mathbb{E}_\mu[|F||L_{v,k}^\phi - L_v^\phi|] \\
&+ \|g_F\|_{\sup}\frac{\sup_{k\in\mathbb{N}}\mathbb{E}_\mu[(L_{v,k}^\phi)^2]}{r} + \|g_F\|_{\sup}\frac{\sup_{k\in\mathbb{N}}\sup_{n\in\mathbb{N}}\mathbb{E}_{\mu^{(N_n)}}[(L_{v,k}^\phi)^2]}{r}.
\end{aligned}
$$

The constants $\sup_{k\in\mathbb{N}}\mathbb{E}_\mu[(L_{v,k}^\phi)^2]$, $\sup_{k\in\mathbb{N}}\sup_{n\in\mathbb{N}}\mathbb{E}_{\mu^{(N_n)}}[(L_{v,k}^\phi)^2]$ are finite due to the estimates for $|J_2|, |J_3|, |J_4|$ in the proof of Lemma 5.5. Now the second identity in (5.16) follows from Lemma 5.5 and the weak convergence $\mu^{(N_n)} \to \mu$ as $n \to \infty$.

Note that

$$
\nabla_v^\Gamma F = \sum_{j=1}^n \partial_j g_F(\langle f_1, \cdot \rangle, \ldots, \langle f_n, \cdot \rangle)\langle (\nabla f_j, v)_{\mathbb{R}^d}, \cdot \rangle.
$$

Thus, showing the first identity in (5.16) is a special case of proving the second one. Just take the monomial $\langle (\nabla f_j, v)_{\mathbb{R}^d}, \cdot \rangle$ instead of the monomial $\langle \operatorname{div} v, \cdot \rangle$ and the function 1 instead of L_v^ϕ, all the other functions involved are bounded and continuous.

Hence we have shown (5.15). The fact that μ is a canonical Gibbs measure now follows from [AKR98b, Theo. 4.3]. ∎

5.3. **Identification of the accumulation points as the distribution of an infinite volume, infinite particle stochastic dynamics.** Let us fix an accumulation point μ of $(\mu^{(N)})_{N\in\mathbb{N}}$. Then for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we consider the bilinear from

$$
\mathcal{E}_\mu(F, G) := \int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma) \rangle_{T_\gamma(\Gamma)} \, d\mu(\gamma)
$$

$$
= \int_\Gamma \sum_{x\in\gamma} (\nabla^\Gamma F(\gamma, x), \nabla^\Gamma G(\gamma, x))_{\mathbb{R}^d} \, d\mu(\gamma), \quad (5.18)
$$

where $\langle \cdot, \cdot \rangle_{T_\gamma(\Gamma)} = \sum_{x\in\gamma}(\cdot, \cdot)_{\mathbb{R}^d}$ is the scalar product in the tangent space $T_\gamma(\Gamma)$, see [AKR98b] for details. Using the integration by parts formula derived in Theorem 5.9 we obtain for $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$:

$$
\mathcal{E}_\mu(F, G) = \int_\Gamma H_\mu FG \, d\mu,
$$

where

$$H_\mu F = -\sum_{i,j=1}^{N} \partial_i \partial_j g_F(\langle f_1, \cdot \rangle, \dots, \langle f_N, \cdot \rangle) \langle (\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \cdot \rangle$$

$$-\sum_{j=1}^{N} \partial_j g_F(\langle f_1, \cdot \rangle, \dots, \langle f_N, \cdot \rangle)\Big(\langle \Delta f_j, \cdot \rangle + L^\phi_{\nabla f_j}\Big) \quad (5.19)$$

for $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ as in (4.8).

Theorem 5.10. *Assume the conditions as in Theorem 5.1. Furthermore, let $\mathbf{P}$ be an accumulation point of $(\mathbf{P}^{(N)})_{N \in \mathbb{N}}$ with invariant canonical Gibbs measure μ provided in Theorem 5.1. Then $\mathbf{P}$ solves the martingale problem for $(-H_\mu, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ with initial distribution μ, i.e., for all $G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$,*

$$G(\mathbf{X}(t)) - G((\mathbf{X}(0)) + \int_0^t H_\mu G((\mathbf{X}(u))\, du, \qquad t \geq 0, \quad (5.20)$$

is an $\mathbf{F}_t$-martingale under $\mathbf{P}$ and $\mathbf{P} \circ \mathbf{X}(0)^{-1} = \mu$.

Proof: For $t, s \geq 0$, we define the following random variable on $C([0, \infty), \Gamma)$:

$$U(\mathbf{X}, t, s) := G(\mathbf{X}(t+s)) - G(\mathbf{X}(t)) + \int_t^{t+s} H_\mu G(\mathbf{X}(u))\, du.$$

Corresponding to

$$G = g_G(\langle f_1, \cdot \rangle, \dots, \langle f_n, \cdot \rangle) \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$$

we define

$$\widetilde{G}(x) := g_G\Big(\sum_{i=1}^{N} f_1(x_i), \dots, \sum_{i=1}^{N} f_n(x_i)\Big), \quad x = (x_1, \dots, x_N) \in \Lambda^N.$$

Note that $G(\mathrm{sym}^{(N)}(\cdot)) = \widetilde{G}$ on $\widetilde{\Lambda^N}$. Since the $f_1, \dots, f_n$ have compact support there exists $N_0 \in \mathbb{N}$ such that $\widetilde{G}$ is an element of $C_c^2((\mathring{\Lambda}_N)^N) \subset D(H_{\Lambda_N, N})$ for all $N \geq N_0$. Hence for $N \geq N_0$, $G \in D(H_{\Lambda_N}^{(N)})$ and we have the pointwise representation of $H_{\Lambda_N}^{(N)}G$ provided in (4.9). Notice that this representation coincides with the pointwise representation of $H_\mu G$, see (5.19).

The trace filtration obtained by restricting $(\mathbf{F}_t)_{t \geq 0}$ to $C([0, \infty), \Gamma_\Lambda^{(N)})$ coincides with the natural filtration of $C([0, \infty), \Gamma_\Lambda^{(N)})$. Furthermore, $\mathbf{P}^{(N)}$ solves the martingale problem for $(-H_{\Lambda_N}^{(N)}, D(H_{\Lambda_N}^{(N)}))$ w.r.t. $(\mathbf{F}_{\Lambda_N}^{(N)}(t))_{t \geq 0}$. Therefore we have for all $\mathbf{F}_t$-measurable, bounded, continuous $F_t : C([0, \infty), \Gamma) \to \mathbb{R}$ and $N \geq N_0$ that $\mathbb{E}_{\mathbf{P}^{(N)}}[F_t U(t, s)] = 0$. Thus it follows that

$$0 = \lim_{N \to \infty} \mathbb{E}_{\mathbf{P}^{(N)}}[F_t U(t, s)]. \quad (5.21)$$

Now let $(N_n)_{n \in \mathbb{N}}$ a subsequence such that $\mathbf{P}^{(N_n)} \to \mathbf{P}$ weakly. Having (5.21) it remains to show

$$\lim_{n \to \infty} \mathbb{E}_{\mathbf{P}^{(N_n)}}[F_t U(t, s)] = \mathbb{E}_{\mathbf{P}}[F_t U(t, s)] \quad (5.22)$$

to obtain (5.20). We have

$$
\begin{aligned}
|\mathbb{E}_{\mathbf{P}^{(N_n)}}&[F_t U(t,s)] - \mathbb{E}_{\mathbf{P}}[F_t U(t,s)]| \\
&\leq |\mathbb{E}_{\mathbf{P}^{(N_n)}}[F_t(G(\mathbf{X}(t+s)) - G(\mathbf{X}(t)))] - \mathbb{E}_{\mathbf{P}}[F_t(G(\mathbf{X}(t+s)) - G(\mathbf{X}(t)))]| \\
&\qquad + \int_t^{t+s} |\mathbb{E}_{\mathbf{P}^{(N_n)}}[F_t H_\mu G(\mathbf{X}(u))] - \mathbb{E}_{\mathbf{P}}[F_t H_\mu G(\mathbf{X}(u))]| \, du. \quad (5.23)
\end{aligned}
$$

The first term on the right hand side of the estimate (5.23) converges to zero as $n \to \infty$, because the function $F_t(G(\mathbf{X}(t+s)) - G(\mathbf{X}(t)))$ is bounded and continuous. Showing that

$$
|\mathbb{E}_{\mathbf{P}^{(N_n)}}[F_t H_\mu G(\mathbf{X}(u))] - \mathbb{E}_{\mathbf{P}}[F_t H_\mu G(\mathbf{X}(u))]| \to 0 \quad \text{as} \quad n \to \infty \quad \forall u \in [t, t+s]
$$

is essentially the same as proving (5.16), done in the proof of Theorem 5.9. Now using the Cauchy–Schwartz inequality, the fact that μ and $\mu^{(N_n)}$ are the invariant measures of $\mathbf{P}$ and $\mathbf{P}^{(N_n)}$, respectively, and the boundedness of $\{\mathbb{E}_{\mu^{(N_n)}}[(H_\mu G)^2] \,|\, n \in \mathbb{N}\}$ we find a constant $C_{18} < \infty$ independent of $u \in [t, t+s]$ and $n \in \mathbb{N}$ such that

$$
|\mathbb{E}_{\mathbf{P}^{(N_n)}}[F_t H_\mu G(\mathbf{X}(u))] - \mathbb{E}_{\mathbf{P}}[F_t H_\mu G(\mathbf{X}(u))]| \leq C_{18}.
$$

Therefore, the second term on the right hand side of the estimate (5.23) converges to zero as $n \to \infty$ by Lebesgue dominated convergence. Thus, (5.22) is shown.

Obviously, we have $\mathbf{P} \circ \mathbf{X}(t)^{-1} = \mu$ for all $t \geq 0$, in particular $\mathbf{P} \circ \mathbf{X}(0)^{-1} = \mu$. ∎

Remark 5.11. Using Itô's formula, Theorem 5.10 implies that each accumulation point $\mathbf{P}$ of $(\mathbf{P}^{(N)})_{N\in\mathbb{N}}$ solves the following infinite system of stochastic differential equation in the sense of the associated martingale problem:

$$
dx(t) = -\beta \sum_{\substack{y(t)\in\mathbf{X}(t) \\ y(t)\neq x(t)}} \nabla\phi(x(t) - y(t)) \, dt + \sqrt{2} \, dB^x(t),
$$

$$
\mathbf{P} \circ \mathbf{X}(0)^{-1} = \mu, \tag{5.24}
$$

where $x(t) \in \mathbf{X}(t) \in \Gamma$, $(B^x)_{x\in\gamma}$, $\gamma \in \Gamma$, is a sequence of independent Brownian motions and μ is the invariant measure corresponding to $\mathbf{P}$.

5.4. **Identification of the accumulation points as Markov processes and uniqueness.** By [AKR98b] the closure of $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ on $L^2(\Gamma, \mu)$, in sequel denoted by $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min}))$, is conservative, local and quasi-regular, hence associated with a diffusion process on Γ. When started with μ its distribution $\mathbf{P}_\mu$ also satisfies the martingale problem (5.20). So far we do not know whether $\mathbf{P}_\mu = \mathbf{P}$ with $\mathbf{P}$ as in Theorem 5.10. A first step to that identification yields the following convergence of the associated Dirichlet forms.

Proposition 5.12. *Let the assumptions in Theorem 5.1 hold and let $(\mu^{(N_n)})_{n\in\mathbb{N}}$ be a subsequence converging to an accumulation point μ of $(\mu^{(N)})_{N\in\mathbb{N}}$. Then for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$*

$$
\lim_{n\to\infty} \mathcal{E}_{\Lambda_{N_n}}^{(N_n)}(F, G) = \mathcal{E}_\mu(F, G). \tag{5.25}
$$

Proof: By polarization identity we can restrict ourself to the case $F = G$. From (5.18) we get that

$$\mathcal{E}_\mu(F, F)$$
$$= \sum_{i,j=1}^n \int_\Gamma \partial_i g_F(\langle f_1, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle) \partial_j g_F(\langle f_1, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle) \langle (\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \gamma \rangle \, d\mu(\gamma)$$

for F as in (4.8). Furthermore, by definition of $\mathcal{E}_\Lambda^{(N)}$, see (4.4), we find

$$\mathcal{E}_{\Lambda_{N_n}}^{(N_n)}(F, F)$$
$$= \sum_{i,j=1}^n \int_\Gamma \partial_i g_F(\langle f_1, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle) \partial_j g_F(\langle f_1, \gamma \rangle, \ldots, \langle f_n, \gamma \rangle) \langle (\nabla f_i, \nabla f_j)_{\mathbb{R}^d}, \gamma \rangle \, d\mu^{(N_n)}(\gamma)$$

again for F as in (4.8). Since $\mu^{(N_n)} \to \mu$ weakly as $n \to \infty$, (5.25) follows by analogous arguments as in the proof of Theorem 5.9. ∎

This convergence, however, is too weak to conclude convergence of the associated semi-groups or resolvents. For this we need the stronger Mosco convergence of quadratic forms. The concepts of Mosco convergence were introduced in [Mos94]. Here we need a generalization of these concepts provided in [KS03].

Definition 5.13. We say that a sequence of Hilbert spaces $(H_n)_{n \in \mathbb{N}}$ converges to a Hilbert space H, if there exists a dense subspace $C \subset H$ and a sequence of operators $(\Phi_n)_{n \in \mathbb{N}}$, where

$$\Phi_n : C \to H_n, \quad n \in \mathbb{N},$$

with the following property:

$$\lim_{n \to \infty} \|\Phi_n u\|_{H_n} = \|u\|_H$$

for all $u \in C$.

Let μ be an accumulation point of $(\mu^{(N)})_{N \in \mathbb{N}}$ and $(\mu^{(N_n)})_{n \in \mathbb{N}}$ a subsequence such that $\lim_{n \to \infty} \mu^{(N_n)} = \mu$. When choosing $C := \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ and the mapping $\Phi_n := R_n$, $n \in \mathbb{N}$, as the choice of the continuous representative of a function from $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \subset L^2(\mu)$ (this can be done uniquely, since μ as a Gibbs measure has full topological support on Γ) and then considering as function in $L^2(\mu^{(N_n)})$, we see that $H_n := L^2(\mu^{(N_n)})$ converges to $H := L^2(\mu)$ in the sense of Definition 5.13 as $n \to \infty$.

Definition 5.14 (strong convergence)**.** Let $(H_n)_{n \in \mathbb{N}}$, $(\Phi_n)_{n \in \mathbb{N}}$, H and C be as in Definition 5.13. We say that a sequence of vectors $(u_n)_{n \in \mathbb{N}}$ with $u_n \in H_n, n \in \mathbb{N}$, converges strongly to a vector $u \in H$, if there exists a sequence $(\tilde{u}_n)_{n \in \mathbb{N}}$ in C with the following properties:

$$\lim_{m \to \infty} \|\tilde{u}_m - u\|_H = 0$$
$$\lim_{m \to \infty} \limsup_{n \to \infty} \|\Phi_n \tilde{u}_m - u_n\|_{H_n} = 0.$$

Definition 5.15 (weak convergence). Let $(H_n)_{n\in\mathbb{N}}$, $(\Phi_n)_{n\in\mathbb{N}}$, H and C be as in Definition 5.13. We say that a sequence of vectors $(u_n)_{n\in\mathbb{N}}$ with $u_n \in H_n, n \in \mathbb{N}$, converges weakly to a vector $u \in H$, if

$$\lim_{n\to\infty}(u_n, v_n)_{H_n} = (u, v)_H$$

for every sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \in H_n$, $n \in \mathbb{N}$, which strongly converges to $v \in H$.

In [Kol04][Lem. 2.7] the following simple criterion for strong convergence has been proved.

Lemma 5.16. Let $(H_n)_{n\in\mathbb{N}}$, $(\Phi_n)_{n\in\mathbb{N}}$, H and C be as in Definition 5.13. A sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in H_n, n \in \mathbb{N}$, converges strongly to a vector $u \in H$, if and only if

$$\lim_{n\to\infty}\|u_n\|_{H_n} = \|u\|_H \quad \text{and} \quad \lim_{n\to\infty}(u_n, \Phi_n(v))_{H_n} = (u, v)_H \quad \text{for all} \quad v \in C.$$

Definition 5.17. Let $(H_n)_{n\in\mathbb{N}}$, $(\Phi_n)_{n\in\mathbb{N}}$, H and C be as in Definition 5.13. We say that a sequence of bounded operators $(B_n)_{n\in\mathbb{N}}$ with $B_n \in L(H_n), n \in \mathbb{N}$, converges strongly to a bounded operator $B \in L(H)$, if for every sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in H_n$, $n \in \mathbb{N}$, which strongly converges to $u \in H$, the sequence $(B_n u_n)_{n\in\mathbb{N}}$ strongly converges to Bu.

Next we consider convergence of quadratic forms $\mathcal{Q}$. Recall that a quadratic form on a Hilbert space $\mathcal{H}$ is given by a bilinear form $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$, where $D(\mathcal{E}) \subset \mathcal{H}$. We consider only densely defined, non-negative, closed, symmetric bilinear forms. Then we define the corresponding quadratic form $\mathcal{Q} : \mathcal{H} \to \overline{\mathbb{R}}$ by setting

$$\mathcal{Q}(u) := \left\{ \begin{array}{ll} \mathcal{E}(u, u) & \text{if} \quad u \in D(\mathcal{E}) \\ \infty & \text{otherwise} \end{array} \right..$$

Recall that closedness of $(\mathcal{E}, D(\mathcal{E}))$ is equivalent to lower semi-continuity of $\mathcal{Q} : \mathcal{H} \to \overline{\mathbb{R}}$.

Definition 5.18 (Mosco convergence). Let $(H_n)_{n\in\mathbb{N}}$, $(\Phi_n)_{n\in\mathbb{N}}$, H and C be as in Definition 5.13. We say that a sequence of quadratic forms $(\mathcal{Q}_n)_{n\in\mathbb{N}}$ with $\mathcal{Q}_n : H_n \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, Mosco converges to a quadratic form $\mathcal{Q} : H \to \overline{\mathbb{R}}$, if the following conditions hold:

(M1) If a sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in H_n, n \in \mathbb{N}$, weakly converges to a vector $u \in H$, then

$$\mathcal{Q}(u) \leq \liminf_{n\to\infty} \mathcal{Q}_n(u_n).$$

(M2) For all $u \in H$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in H_n, n \in \mathbb{N}$, which strongly converges to u and

$$\mathcal{Q}(u) = \lim_{n\to\infty} \mathcal{Q}_n(u_n).$$

In [Mos94] it is proved that Mosco convergence of a sequence of quadratic forms is equivalent to the convergence, in the strong operator sense, of the sequence of semigroups and resolvents, respectively, associated with the corresponding bilinear forms. In [KS03] this result is generalized to the present situation, where we have a sequence of Hilbert spaces. Here strong convergence of bounded operators has to be understood in the sense of Definition 5.17.

We are interested in the case where $\mathcal{Q}_n$ is the quadratic form corresponding to $(\mathcal{E}_{\Lambda_{N_n}}^{(N_n)}, D(\mathcal{E}_{\Lambda_{N_n}}^{(N_n)}))$, $n \in \mathbb{N}$, and $\mathcal{Q}$ the quadratic form corresponding to $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min}))$. In order

to check (M1) we need to consider a closed extension of $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ on $L^2(\Gamma, \mu)$, which possibly is larger than $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min}))$. Let $\mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ be the set of all maps defined as follows:

$$\Gamma \ni \gamma \mapsto \sum_{i=1}^N F_i(\gamma)v_i,$$

where $F_1, \ldots, F_N \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ and $v_1, \ldots, v_N \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$. For $V = \sum_{i=1}^N F_i v_i \in \mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we define

$$\mathrm{div}_\phi^\Gamma V := \sum_{i=1}^N (\nabla_{v_i}^\Gamma F_i + B_{v_i}^\phi F_i v_i),$$

see (5.13) and Lemma 5.6. From the integration by parts formula (5.15) provided in Theorem 5.9 we can conclude that for all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ and $V \in \mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$

$$\int_\Gamma \langle \nabla^\Gamma F, V \rangle_{T\Gamma} \, d\mu = - \int_\Gamma F \mathrm{div}_\phi^\Gamma V \, d\mu.$$

Let $((\mathrm{div}_\phi^\Gamma)^*, D((\mathrm{div}_\phi^\Gamma)^*))$ denote the adjoint of $(\mathrm{div}_\phi^\Gamma, \mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ as an operator from $L^2(\Gamma, T\Gamma, \mu)$ to $L^2(\Gamma, \mu)$. By definition, $G \in L^2(\Gamma, \mu)$ belongs to $D((\mathrm{div}_\phi^\Gamma)^*)$ if and only if there exist a unique $(\mathrm{div}_\phi^\Gamma)^* G \in L^2(\Gamma, T\Gamma, \mu)$ such that

$$\int_\Gamma G \, \mathrm{div}_\phi^\Gamma V \, d\mu = - \int_\Gamma \langle (\mathrm{div}_\phi^\Gamma)^* G, V \rangle_{T\Gamma} \, d\mu \quad \text{for all} \quad V \in \mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma).$$

We set $D(\mathcal{E}_\mu^{\max}) := D((\mathrm{div}_\phi^\Gamma)^*)$, $d^\mu := (\mathrm{div}_\phi^\Gamma)^*$ and define

$$\mathcal{E}_\mu^{\max}(F, G) := \int_\Gamma \langle d^\mu F, d^\mu G \rangle_{T\Gamma} \, d\mu, \quad \text{for all} \quad F, G \in D(\mathcal{E}_\mu^{\max}).$$

From the integration by part formula (5.15) it follows that

$$\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \subset D(\mathcal{E}_\mu^{\max}) \quad \text{and} \quad d^\mu = \nabla^\Gamma \quad \text{on} \quad \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma).$$

Hence the densely defined, non-negative, closed, symmetric bilinear form $(\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$ extends $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min}))$. From general theory it is clear that $(\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$ has an associated self-adjoint generator $(H_\mu^{\max}, D(H_\mu^{\max}))$. However, it is not clear whether it is Markovian. In [SS03] it is shown that for non-negative interaction potentials ϕ the generator of the closure of $\mathcal{E}_\mu^{\max}$ restricted to $D(\mathcal{E}_\mu^{\max}) \cap L^\infty(\mu)$ is the maximum Markovian self-adjoint extension of $(H_\mu, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$.

Condition (M1) we can only check, when $\mathcal{Q}$ is the quadratic form corresponding to $(\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$. Condition (M2) we can only check, when $\mathcal{Q}$ is the quadratic form corresponding to $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min}))$. Hence we have to assume that $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min})) = (\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$. In the case that $(H_\mu^{\max}, D(H_\mu^{\max}))$ is Markovian, this is equivalent to the so-called Markov uniqueness property, see e.g. [Ebe99]. Obviously, the property $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min})) = (\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$ is weaker than essential self-adjointness of $(H_\mu, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$.

To verify (M1) we need strong convergence of the logarithmic derivatives.

Proposition 5.19. Let the conditions in Theorem 5.1 hold and $V \in \mathcal{V}FC_b^\infty(\mathcal{D}, \Gamma)$. Then $\operatorname{div}_\phi^\Gamma V$ considered as an element in $L^2(\mu^{(N_n)})$ converges strongly to $\operatorname{div}_\phi^\Gamma V$ considered as an element in $L^2(\mu)$ as $n \to \infty$ (recall that $\operatorname{div}_\phi^\Gamma V$ is a pointwise defined function on Γ).

Proof: In the proof of Theorem 5.9 we have shown that

$$\lim_{n\to\infty} \int_{\Gamma^{(N_n)}_{\Lambda_{N_n}}} F \operatorname{div}_\phi^\Gamma V \, d\mu^{(N_n)} = \int_\Gamma F \operatorname{div}_\phi^\Gamma V \, d\mu \tag{5.26}$$

for all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$. Hence, by Lemma 5.16 it remains to show that

$$\lim_{n\to\infty} \int_{\Gamma^{(N_n)}_{\Lambda_{N_n}}} (\operatorname{div}_\phi^\Gamma V)^2 \, d\mu^{(N_n)} = \int_\Gamma (\operatorname{div}_\phi^\Gamma V)^2 \, d\mu. \tag{5.27}$$

Since for $V = \sum_{i=1}^N F_i v_i \in \mathcal{V}FC_b^\infty(\mathcal{D}, \Gamma)$ we have

$$\operatorname{div}_\phi^\Gamma V = \sum_{i=1}^N (\nabla_{v_i}^\Gamma F_i + \langle \operatorname{div} v_i, \cdot \rangle + L_{v_i}^\phi F_i v_i),$$

(5.27) follows from

$$\lim_{n\to\infty} \int_{\Gamma^{(N_n)}_{\Lambda_{N_n}}} |\langle \operatorname{div} w, \cdot \rangle L_v^\phi| \, d\mu^{(N_n)} = \int_\Gamma |\langle \operatorname{div} w, \cdot \rangle L_v^\phi| \, d\mu \tag{5.28}$$

$$\text{and} \quad \lim_{n\to\infty} \int_{\Gamma^{(N_n)}_{\Lambda_{N_n}}} (L_v^\phi)^2 \, d\mu^{(N_n)} = \int_\Gamma (L_v^\phi)^2 \, d\mu \tag{5.29}$$

for all $v, w \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$. For all the other terms convergence can be shown as convergence of (5.26). Using the notation as in the proof of Lemma 5.5 we get:

$$|\mathbb{E}_{\mu^{(N_n)}}[(L_v^\phi)^2] - \mathbb{E}_\mu[(L_v^\phi)^2]| \leq |\mathbb{E}_{\mu^{(N_n)}}[(L_v^\phi)^2 - (L_{v,k}^\phi)^2]| + \mathbb{E}_{\mu^{(N_n)}}[(L_{v,k}^\phi)^2 - ((L_{v,k}^\phi)^2)_r]$$

$$+ |\mathbb{E}_{\mu^{(N_n)}}[((L_{v,k}^\phi)^2)_r] - \mathbb{E}_\mu[((L_{v,k}^\phi)^2)_r]| + \mathbb{E}_\mu[(L_{v,k}^\phi)^2 - ((L_{v,k}^\phi)^2)_r] + |\mathbb{E}_\mu[(L_{v,k}^\phi)^2 - (L_v^\phi)^2]|$$

$$\leq \left(\sup_{n\in\mathbb{N}} \sqrt{\mathbb{E}_{\mu^{(N_n)}}[(L_v^\phi)^2]} + \sup_{k\in\mathbb{N}} \sup_{n\in\mathbb{N}} \sqrt{\mathbb{E}_{\mu^{(N_n)}}[(L_{v,k}^\phi)^2]} \right) \sqrt{C_{16}(k)}$$

$$+ \left(\sqrt{\mathbb{E}_\mu[(L_v^\phi)^2]} + \sup_{k\in\mathbb{N}} \sqrt{\mathbb{E}_\mu[(L_{v,k}^\phi)^2]} \right) \sqrt{\mathbb{E}_\mu[(L_v^\phi - L_{v,k}^\phi)^2]}$$

$$+ |\mathbb{E}_{\mu^{(N_n)}}[((L_{v,k}^\phi)^2)_r] - \mathbb{E}_\mu[((L_{v,k}^\phi)^2)_r]| + \frac{\sup_{k\in\mathbb{N}} \mathbb{E}_\mu[|L_{v,k}^\phi|^3]}{r}$$

$$+ \frac{\sup_{k\in\mathbb{N}} \sup_{n\in\mathbb{N}} \mathbb{E}_{\mu^{(N_n)}}[|L_{v,k}^\phi|^3]}{r}.$$

The constants $\sup_{k\in\mathbb{N}} \mathbb{E}_\mu[|L_{v,k}^\phi|^3]$, $\sup_{k\in\mathbb{N}} \sup_{n\in\mathbb{N}} \mathbb{E}_{\mu^{(N_n)}}[|L_{v,k}^\phi|^3]$ are finite due to Lemma 5.7. Furthermore, the constants $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mu^{(N_n)}}[(L_v^\phi)^2]$, $\sup_{k\in\mathbb{N}} \sup_{n\in\mathbb{N}} \mathbb{E}_{\mu^{(N_n)}}[(L_{v,k}^\phi)^2]$ and $\sup_{k\in\mathbb{N}} \mathbb{E}_\mu[(L_{v,k}^\phi)^2]$ are finite due to the estimates for $|J_2|, |J_3|, |J_4|$ provided in the proof of Lemma 5.5. Now (5.29) follows from Lemma 5.5 and the weak convergence $\mu^{(N_n)} \to \mu$ as $n \to \infty$. (5.28) can be shown analogously, since the constant $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mu^{(N_n)}}[|\langle \operatorname{div} w, \cdot \rangle|^p]$

is finite for all $1 \leq p < \infty$. For $p = 2$ this is shown in the proof of Theorem 5.9, see (5.17). The proof easily generalizes to all $1 \leq p < \infty$ due to the Ruelle bound (3.3). ∎

Theorem 5.20. *Let the assumptions in Theorem 5.1 hold and let $(\mu^{(N_n)})_{n\in\mathbb{N}}$ be a subsequence converging to an accumulation point μ of $(\mu^{(N)})_{N\in\mathbb{N}}$. Suppose that $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min}))$ $= (\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$. Then the sequence of quadratic forms corresponding to $(\mathcal{E}_{\Lambda_{N_n}}^{(N_n)}, D(\mathcal{E}_{\Lambda_{N_n}}^{(N_n)}))_{n\in\mathbb{N}}$ Mosco converges to the quadratic form corresponding to $(\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$.*

Proof: Since $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min}))$ is the closure of $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$, Proposition 5.12 together with [Kol04][Lem. 2.8] implies (M2)

In order to check condition (M1), we consider a sequence $(F_n)_{n\in\mathbb{N}}$ with $F_n \in L^2(\mu^{(N_n)})$, $n \in \mathbb{N}$, which weakly converges to $F \in L^2(\mu)$. Furthermore, recall that in Proposition 5.19 we have shown strong convergence of $\operatorname{div}_\phi^\Gamma V$ considered as an element in $L^2(\mu^{(N_n)})$ to $\operatorname{div}_\phi^\Gamma V$ considered as an element in $L^2(\mu)$ as $n \to \infty$ for all $V \in \mathcal{V}FC_b^\infty(\mathcal{D}, \Gamma)$.

First let us assume that $F \in D(\mathcal{E}_\mu^{\max})$. Then

$$\left(\int_\Gamma \langle d^\mu F, V \rangle_{T\Gamma}\, d\mu \right)^2 = \left(\int_\Gamma F \operatorname{div}_\phi^\Gamma V\, d\mu \right)^2 = \lim_{n\to\infty} \left(\int_{\Gamma_{\Lambda_{N_n}}^{(N_n)}} F_n \operatorname{div}_\phi^\Gamma V\, d\mu^{(N_n)} \right)^2$$

$$\leq \liminf_{n\to\infty} \mathcal{Q}_n(F_n) \int_\Gamma \langle V, V \rangle_{T\Gamma}\, d\mu \quad \text{for all} \quad V \in \mathcal{V}FC_b^\infty(\mathcal{D}, \Gamma), \quad (5.30)$$

where $\mathcal{Q}_n$ is the quadratic form corresponding to $(\mathcal{E}_{\Lambda_{N_n}}^{(N_n)}, D(\mathcal{E}_{\Lambda_{N_n}}^{(N_n)}))$. Since $\mathcal{V}FC_b^\infty(\mathcal{D}, \Gamma)$ is dense in $L^2(\Gamma, T\Gamma, \mu)$, (5.30) yields (M1) for $F \in D(\mathcal{E}_\mu^{\max})$.

For $F \notin D(\mathcal{E}_\mu^{\max})$ we have $\mathcal{Q}(F) = \infty$, where $\mathcal{Q}$ is the quadratic form corresponding to $(\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$. We assume that $\liminf_{n\to\infty} \mathcal{Q}_n(F_n) = C_{19} < \infty$, then

$$\left| \int_\Gamma F \operatorname{div}_\phi^\Gamma V\, d\mu \right| \leq \sqrt{C_{19}} \|V\|_{L^2(\Gamma, T\Gamma, \mu)} \quad \text{for all} \quad V \in \mathcal{V}FC_b^\infty(\mathcal{D}, \Gamma).$$

I.e., $F \in (\operatorname{div}_\phi^\Gamma)^* = D(\mathcal{E}_\mu^{\max})$. That is a contradiction! Hence $\liminf_{n\to\infty} \mathcal{Q}_n(F_n) = \infty$. ∎

Remark 5.21. *The essential ideas for proving Theorem 5.20 we got from the proof of [Kol04][Prop. 4.1].*

We denote the strongly continuous contraction semi-group associated with $(\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$ by $(T_\mu(t))_{t\geq 0}$.

Corollary 5.22. *Let the assumptions in Theorem 5.1 hold and let $(\mu^{(N_n)})_{n\in\mathbb{N}}$ be a subsequence converging to an accumulation point μ of $(\mu^{(N)})_{N\in\mathbb{N}}$. Suppose that $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min}))$ $= (\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$. Then the sequence of semi-groups $(T_{\Lambda_{N_n}}^{(N_n)}(t))$ strongly converges to $T_\mu(t)$ as $n \to \infty$ for all $t \geq 0$. The same holds for the corresponding resolvents.*

Proof: By Theorem 5.20 this follows directly from [KS03, Theo. 2.4]. ∎

Theorem 5.23. *Let the assumptions in Theorem 5.1 hold and let* $\mathbf{P}$ *be an accumulation point of* $(\mathbf{P}^{(N)})_{N \in \mathbb{N}}$ *with invariant canonical Gibbs measure* μ. *Suppose that* $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min})) = (\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$. *Then* $\mathbf{P}$ *is the law of a Markov process with initial distribution* μ *and semi-group* $(T_\mu(t))_{t \geq 0}$. *In particular, all accumulation points of* $(\mathbf{P}^{(N)})_{N \in \mathbb{N}}$ *with the same invariant measure* μ *coincide.*

Proof: Let $(\mathbf{P}^{(N_n)})_{n \in \mathbb{N}}$ be a subsequence such that $\lim_{n \to \infty} \mathbf{P}^{(N_n)} = \mathbf{P}$. This implies $\lim_{n \to \infty} \mu^{(N_n)} = \mu$. From Corollary 5.22 we now can conclude that $T_{\Lambda_{N_n}}^{(N_n)}(t)$ converges strongly to $T_\mu(t)$ as $n \to \infty$ for all $t \geq 0$. Thus, finite dimensional distributions of $\mathbf{P}$ are given through $(T_\mu(t))_{t \geq 0}$. Since this holds for all accumulation points of $(\mathbf{P}^{(N)})_{N \in \mathbb{N}}$ with invariant measure μ, they all coincide. ∎

6. Application to the problem of equivalence of ensembles

Grand canonical Gibbs measures correspond to an interaction potential ϕ, inverse temperature $\beta \geq 0$ and activity function $z \geq 0$. An interesting question is the equivalence of the grand canonical and canonical ensemble, i.e., the question whether grand canonical and canonical Gibbs measures corresponding to an interaction potential ϕ and inverse temperature β coincide for a certain relation between their activity function z and particle density ρ, respectively, see e.g. [Geo79, Chap. 6]. Furthermore, it is of interest whether one can approximate grand canonical Gibbs measures by finite volume canonical Gibbs measures.

Theorem 6.1. *Assume that the conditions in Theorem 5.1 hold and that we are in the low density, high temperature regime, i.e.,*

$$\rho < \frac{1}{2 \exp(2\beta K + 1) J(\beta)}.$$

Then the sequence of finite volume canonical Gibbs measures with empty boundary condition $(\mu^{(N)})_{N \in \mathbb{N}}$ *converges to a canonical Gibbs measure* μ *with constant density* ρ *as* $N \to \infty$. *Furthermore,* μ *is a grand canonical Gibbs measure corresponding to the activity*

$$z = \lim_{N \to \infty} N \frac{Z_{\Lambda_N}^{(N-1)}}{Z_{\Lambda_N}^{(N)}}. \tag{6.1}$$

Proof: Let us fix an accumulation point μ of $(\mu^{(N)})_{N \in \mathbb{N}}$, i.e., there exists a subsequence $(\mu^{(N_m)})_{m \in \mathbb{N}}$ such that $\mu^{(N_m)} \to \mu$ weakly as $m \to \infty$. From Theorem 5.9 we know that μ is a canonical Gibbs measure. Let $f \in C_c(\mathbb{R}^d)$. Then $\langle f, \cdot \rangle$ is a continuous function on $(\Gamma, d_{(\beta/3)\Phi, h})$ and as in the proof of Theorem 5.9 we can show that

$$\mathbb{E}_\mu[\langle f, \cdot \rangle] = \lim_{m \to \infty} \mathbb{E}_{\mu^{(N_m)}}[\langle f, \cdot \rangle].$$

Now (3.2) yields

$$\mathbb{E}_\mu[\langle f, \cdot \rangle] = \lim_{m \to \infty} \int_{\Lambda_{N_m}} f(x) \, k_{\Lambda_{N_m}}^{(1, N_m)}(x) \, dx_{\Lambda_{N_m}}.$$

In [BPK70, Sect. 4] it is proved that in the low density, high temperature regime

$$\lim_{m \to \infty} k_{\Lambda_{N_m}}^{(1, N_m)}(x) = \rho \quad \text{for all} \quad x \in \mathbb{R}^d.$$

Hence, using the Ruelle bound (3.3) and Lebesgue's dominated convergence theorem, we obtain

$$\mathbb{E}_\mu[\langle f, \cdot \rangle] = \rho \int_{\mathbb{R}^d} f \, dx. \tag{6.2}$$

Thus, μ has constant density ρ.

Moreover, in [BPK70, Sect. 4] it is proved that in the low density, high temperature regime there exits a sequence $(k^{(n)})_{n \in \mathbb{N}}$ of functions $k^{(n)} : \mathbb{R}^{n \cdot d} \to \mathbb{R}$ such that

$$\lim_{m \to \infty} k^{(n, N_m)}_{\Lambda_{N_m}}(x_1, \ldots, x_n) = k^{(n)}(x_1, \ldots, x_n) \text{ for all } (x_1, \ldots, x_n) \in \mathbb{R}^{n \cdot d} \text{ and all } n \in \mathbb{N}.$$

Using the Ruelle bound, see Theorem 3.2, and analogous arguments as in the derivation of (6.2), we can identify the sequence $(k^{(n)})_{n \in \mathbb{N}}$ as the correlation functions of μ. Since this is true for all accumulation points of $(\mu^{(N)})_{N \in \mathbb{N}}$, all accumulation points coincide and $(\mu^{(N)})_{N \in \mathbb{N}}$ converges to the canonical Gibbs measure μ as $N \to \infty$.

Finally, [BPK70, Theo. I] together with [BPK70, Theo. IV] implies that the sequence of correlation functions $(k^{(n)})_{n \in \mathbb{N}}$ fulfills the Kirkwood–Salsburg equations for z as given in (6.1). Thus, μ is a grand canonical Gibbs measure corresponding to ϕ, β and z. ∎

Remark 6.2. A related result has been proved in [Geo95]. There the author derived an approximation of grand canonical Gibbs measures by finite volume micro canonical Gibbs measures with periodic boundary condition. It seems to be quite feasible to adapt the proof to the canonical case. However, since that proof heavily relays on the choice of a periodic boundary condition, it still would not cover our case (empty boundary condition).

Corollary 6.3. *Assume that the conditions in Theorem 5.1 hold and that we are in the low density, high temperature regime. Let $\mu = \lim_{N \to \infty} \mu^{(N)}$ and suppose that $(\mathcal{E}_\mu^{\min}, D(\mathcal{E}_\mu^{\min})) = (\mathcal{E}_\mu^{\max}, D(\mathcal{E}_\mu^{\max}))$. Then the sequence $(\mathbf{P}^{(N)})_{N \in \mathbb{N}}$ converges in law to a Markov process $\mathbf{P}$ with initial distribution μ and semi-group $(T_\mu(t))_{t \geq 0}$. Furthermore, $\mathbf{P}$ solves the martingale problem for $(-H_\mu, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ with initial distribution μ.*

Proof: This is an immediate consequence of Theorem 5.23 together with Theorem 6.1 and Theorem 5.10. ∎

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